Optimal Design of Batch-Storage Network with Uncertainty and Waste Treatments

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The aim of this study was to find an analytic solution to the problem of determining the optimal capacity of a batch-storage network to meet the demand for finished products in a system undergoing random failures of operating time and/or batch material. The superstructure of the plant considered here consists of a network of serially and/or parallel interlinked batch processes and storage units. The production processes transform a set of feedstock materials into another set of products with constant conversion factors. The final product demand flow is susceptible not only to short-term random variations in the cycle time and batch size but also to long-term variations in the average trend. Some of the production processes have random variations in product quantity. The spoiled materials are treated through regeneration or waste disposal processes. All other processes have random variations only in the cycle time. The objective function of the optimization is minimizing the total cost, which is composed of setup and inventory holding costs as well as the capital costs of constructing processes and storage units. A novel production and inventory analysis, the PSW (periodic square wave) model, provides a judicious graphical method to find the upper and lower bounds of random flows. The advantage of this model is that it provides a set of simple analytic solutions while also maintaining a realistic description of the random material flows between processes and storage units; as a consequence of these analytic solutions, the computation burden is substantially reduced. The proposed method has the potential to rapidly provide very useful data on which to base investment decisions during the early plant design stage. It should be of particular use when these decisions must be made in a highly uncertain business environment. © 2006 American Institute of Chemical Engineers *AIChE J,* 52: 3473–3490, 2006
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Introduction

One of the key issues in supply chain optimization is how to handle uncertainties. The sources of uncertainties in business are ubiquitous; they include product demand forecasting, production equipment malfunctions, off-spec materials, changes in the price of raw materials or finished products, and raw material supply shortages.[1] The uncertainties in product demand forecasting are usually classified into three regimes: (1) long-term trend, (2) mid-term seasonality, and (3) short-term randomness. It is very important to accurately and repeatedly predict the long-term trend and mid-term seasonality because there is sufficient time to adjust the system in response to

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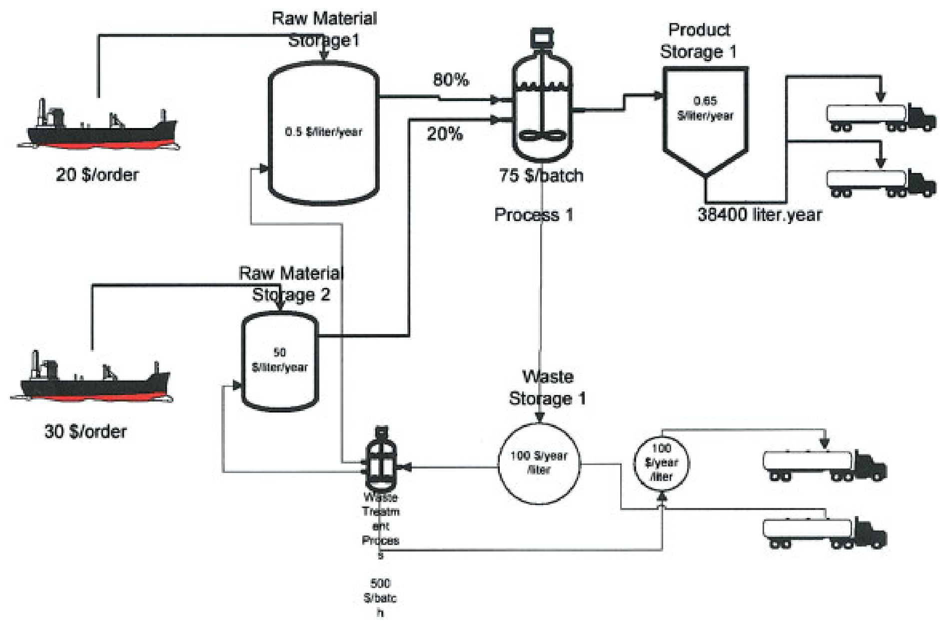

Figure 1. A motivating example plant design problem.

[Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com]

changes in the predictions. However, it is impossible to accurately predict and quickly respond to short-term random variations. Effective inventory management—taking into consideration probabilistic characteristics—is a well-known approach to handling short-term randomness. For example, the safety inventory should be increased in response to increased uncertainty in demand forecasting.[2] The uncertainty in production processes usually has a short-term random component. To minimize equipment malfunctions, an effective maintenance system that includes both preventive and corrective actions is required. When the generation of off-spec materials is unavoidable, a waste treatment system should be considered at the plant design stage. Common waste treatment methods are disposal, regeneration, mixing with high-quality material, and being sold as low-grade material. In refinery plants, for example, oil leaking (slop oil) from equipment is collected into a separate storage unit and then added to the crude storage unit. Another example is found in polyolefin processes, which produce many off-spec materials during grade transitions; some of these off-spec materials are diluted into on-spec materials in blenders or sold as low-grade materials. Uncertainties in raw material supply can be hedged by vendor diversification and/or advanced financial instruments such as forwards and options.[3]

Supply chain optimization under uncertainty has been a subject of intense interest in process system engineering research.[4] Previous research in this area can be classified into three categories[5]: (1) the scenario-based approach, (2) Gaussian quadrature with flexibility index, and (3) Monte Carlo sampling. A common drawback of all of these methods is huge computational times, making them difficult to implement in real applications. Various efforts have been made to reduce the computation time associated with dealing with uncertainty. For example, a convex mixed-integer nonlinear programming (MINLP) model was developed based on the assumption of normality in demand uncertainty,[6] a fuzzy optimization model for the case of unknown probability distribution was proposed,[7] and a convex MINLP model applicable to a linear problem with known bounds of uncertain parameters was suggested.[8] Moreover, a suboptimal approach to sequentially solve the Hamilton–Jacobi–Bellman equation—which is the solution of a dynamic optimization problem, by using Markov chains—was recommended.[9] However, the problem of dealing with uncertainty is not simply a topic of academic research; it is a current and significant problem in the real world. With a view to creating a practical field technique to estimate process reliability, a method based on failure modes and effects analysis (FMEA) was introduced.[10] This technique is composed of a unique definition of uncertainty and a set of simple equations. The fact that this technique effectively treats uncertainty in real problems without depending on advanced mathematical formalisms and with negligible computation suggests that it could represent an exciting new research direction. In the present study, we introduce a novel optimization model resulting in simple analytical solutions with negligible computational burden.

The plant structure is composed of a batch-storage network that can cover most supply chain components, such as raw material purchasing, production, transportation, and finished product demand. The optimal design of a batch-storage network was previously studied using the periodic square wave (PSW) model.[11-15] In the present study, waste treatment processes are added to the network. As shown in Figure 1, off-spec materials produced from failed batches are collected into a storage unit and then, depending on economic factors, are either regenerated into feedstock materials through a process or disposed of to outside sinks.

The processes in this study will be classified into three types according to their random characteristics: (1) processes that possess uncertainty only in the operating time; (2) processes that possess uncertain product batch quantity because of random quantities of off-spec materials; and (3) processes that possess uncertainty in both the operating time and batch quantity. Considering the uncertainties in customer order time and

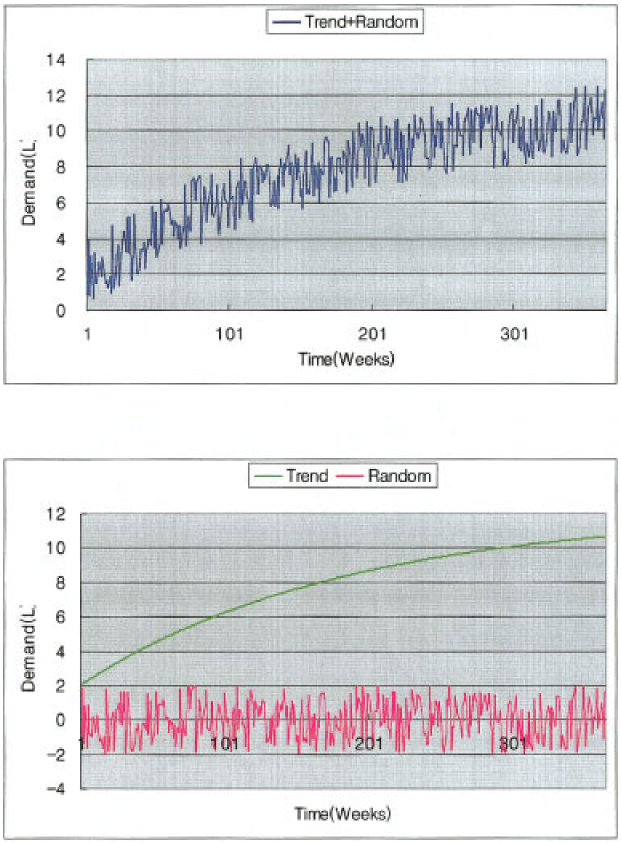

Figure 2. Trend and random component of product demand.

[Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com]

quantity, product demand corresponds to the third type of process. The processes accompanying waste treatment processes fall into the second category of processes, whereas raw material purchase, transportation, and some parts of production processes fall into the first category. Most processes have uncertainties in both operating time and batch quantity but, considering the complexity of handling joint uncertainties of operating time and batch quantity, we exclude such joint uncertainties from the present research, with the exception that product demand has joint uncertainties as given parameters.

In this study, we divide the demand variation into long-term trend and short-term randomness, as can be seen from Figure 2. The mid-term seasonality can be considered as part of the long-term trend or can be incorporated into the model by decomposing the periodic signal into a sum of periodic square waves. The long-term trend in demand is averaged monthly or yearly and thus the demand in the near future can be predicted with relatively good accuracy. The prediction error of the long-term trend in the far future will be reduced as it approaches near future. For more than half a century, chemical plants have used a multiperiod formulation to accommodate the long-term trend into the production plan. In this study, we will extend the PSW model to the multiperiod formulation. Note that the multiperiod formulation is nonperiodic, which goes against the basic assumption of the PSW model. We assume that all parameters are constant within a period but vary nonperiodically over multiple periods. Short-term random variations of processes will be treated using the PSW model and a judicious graphical method.

The great advantage of using the PSW model to deal with uncertainty lies in its being based on analytical solutions—which means that this model can be used at negligible computational cost. Thus, the overall computation time is determined by the time required to compute the average flow rates through the network using the multiperiod formulation. The computation time of the multiperiod formulation proposed in this study is about sevenfold that of the linear programming problem of the same size.[14] Price changes are another important source of business uncertainty. A deterministic model to account for financial transactions and cash flows based on the PSW model has been developed.[14] This model in this study can be used to handle long-term price changes; short-term random price changes are neglected in the present work, but will be explored in future research. Note that the processes in the batch-storage network not only are restricted to batch processes but also are extended to semicontinuous and transportation processes.[15]

Problem Definition for the Single Period Problem

We follow the definitions of parameters and variables used in the deterministic version of the model.[13] A chemical plant, which converts raw materials into final products through multiple physicochemical processing steps, is composed of a set of storage units (J) and a set of batch processes (I). Note that storage index $j \in J$ is written as a superscript, whereas process index $i \in I$ is written as a subscript. We assume that one storage unit stores one material and therefore the storage index j corresponds to the material index. Transportation processes are considered as a subset of batch processes without loss of generality. Each storage unit is involved in five types of material movement: purchasing from suppliers [$k \in K(j)$], shipping in response to consumer demand [$m \in M(j)$], discharging to waste disposal sinks [$n \in N(j)$], feeding to production processes, and producing from production processes.

Here, we will consider three types of processes prone to random failures. Figure 3 shows the flow of the type 1 process, which includes only random operating time loss or random increase of idle time, but does not include batch material loss or batch size change. The cycle times of type 1 processes are random variables, which are denoted as bold character $\boldsymbol{\omega}_{(l)}$, where ($l$) is the order of the batch and $\bar{B}$ is the batch size. Figure 4 shows a typical configuration of a type 2 process and Figure 5 shows the flows of that process. Assume that the cycle time of a type 2 process is deterministic, $\boldsymbol{\omega}_{i_2} \equiv \bar{\omega}_{i_2}$. The feed flow to a type 2 process is assumed to have no uncertainty, that is, it has deterministic cycle time $\bar{\omega}_{i_2}$ and batch size $\bar{B}_{i_2}$. However, the batch size of the discharging flows from a type 2

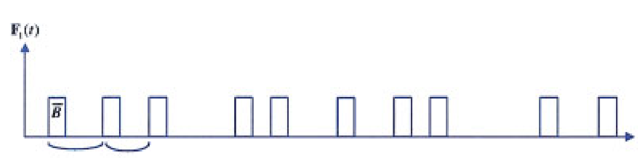

Figure 3. Flow of type 1 process.

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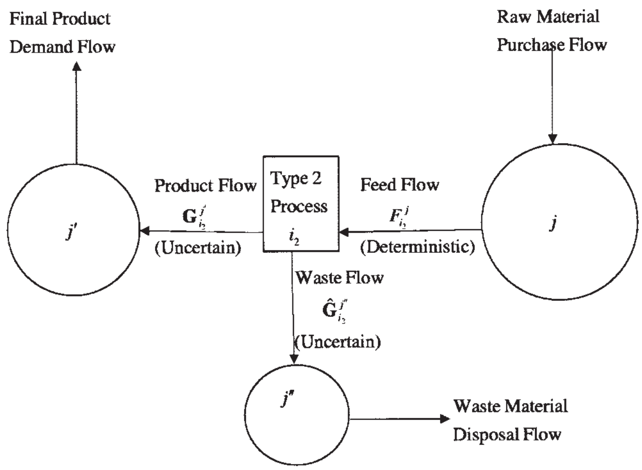

Figure 4. Type 2 process.

process is random. (Assume for simplicity that the processing does not change the material density.) The successful discharging batch quantity $\mathbf{B}'_{i_2(l)}$ goes to a product storage unit and the failed discharging batch quantity $\bar{B}_{i_2} - \mathbf{B}'_{i_2(l)}$ goes to a waste storage unit, as shown in Figure 5. A completely failed batch in a type 2 process consumes not only feedstock materials but also production time equivalent to the processing cycle time.

In summary, in the formalism developed here we assume that the batch size of a type 1 process and the cycle time of a type 2 process are not random, although they are unknown. In modern society, batch material losses associated with raw material purchasing and transportation processes occur very infrequently; therefore, these processes are exclusively considered as type 1 processes in the present study. Production processes may be either type 1 or type 2 processes. Mixing or blending processes do not usually involve batch material loss and thus correspond to type 1 processes. Batch material losses, however, do occur in many reaction processes; thus these processes correspond to type 2 processes. In this study, the element and set of type 1 production processes will be denoted by $i_1 \in I_1$ and the variables or parameters related to type 1 production processes will have subscript i_1. The element and set of type 2 production processes will be denoted by $i_2 \in I_2$ and the variables or parameters related to type 2 production process will have subscript i_2. Note that $I_i \cup I_2 = I$ and $I_i \cap I_2 = \varnothing$. Each production process requires multiple feedstock materials of fixed composition ($f^j_{i_1}$ or $f^j_{i_2}$) and produces multiple products with fixed product yield ($g^j_{i_1}$ or $g^j_{i_2}$). For type 2 processes, the spoiled material from a failed batch goes to storage units other than those storage units where the product goes, according to the fixed waste yield ($\hat{g}^j_{i_2}$). Note that, for the same storage unit j and process i_2, $g^j_{i_2}\hat{g}^j_{i_2} = 0$.

When there is no failure, the material flow from process to storage (or from storage to process) is represented by the PSW model.[13] Each production process is supposed to produce a batch of product during every cycle time ω_i. The cycle time of a production unit is composed of feedstock feeding time (${}^{\backprime}x_i\omega_i$), processing time ($[1 - x'_i\ {}^{\backprime}x_i]\omega_i$), and product discharging time ($x'_i\omega_i$), where $0 \le x'_i, {}^{\backprime}x_i \le 1$ are the storage operation time fraction. Note that the back prime on the variable indicates that the variable is defined for the feeding flow to a production process and the prime on the variable indicates that the variable is defined for the discharging flow from a production process. The processing is initiated at the start-up time ${}^{\backprime}t_i$ (or t'_i). Therefore, the deterministic material flow representation of the PSW model for a production process is composed of four variables: the batch size B_i, the cycle time ω_i, the storage operation time fraction ${}^{\backprime}x_i$ (or x'_i), and the start-up time ${}^{\backprime}t_i$ (or t'_i). The deterministic material flows of raw material purchased, waste disposal, and finished product demand are also represented by four variables B^j_k, ω^j_k, x^j_k, t^j_k, B^j_n, ω^j_n, x^j_n, t^j_n, and B^j_m, ω^j_m, x^j_m, t^j_m, respectively. Note that ω_{i_1}, ω^j_k, ω^j_n, ω^j_m, and B_{i_2} will be considered as random variables. For convenience of presentation, the variables without superscript and subscript—B, ω, and x—will be used to represent the batch size, cycle time, and storage operation time fraction of any process in raw material purchase, production, or finished product demand.

Suppose that the $\boldsymbol{\omega}_{(l)}$ terms in Figure 3 have identical independent distribution functions with respect to (l). $\bar{\omega}$, which is the mean of $\boldsymbol{\omega}_{(l)}$, and $\bar{B}$ are unknowns. For a given convergence limit $0 < \varepsilon_1 \ll 1$ and confidence level $0 < \delta_1 \ll 1$, the weak law of large numbers says that there exists an integer η such that

$$P\left\{\left|\frac{1}{\eta}\sum_{l=1}^{\eta}\boldsymbol{\omega}_{(l)} - \bar{\omega}\right| < \varepsilon_1\right\} \ge 1 - \delta_1$$

From Tchebycheff's inequality, $\eta \ge \mathrm{Var}(\boldsymbol{\omega})/\delta_1\varepsilon_1^2$, that is, $\eta = \mathrm{int}[\mathrm{Var}(\boldsymbol{\omega})/\delta_1\varepsilon_1^2\ ] + 1$ if the least integer is chosen.[16] If we define $\underline{\underline{\omega}}$ as an actual (deterministic) operating time so that $\underline{\underline{\omega}} \le \boldsymbol{\omega}_{(l)}$, then $\underline{\underline{\omega}} = \alpha\bar{\omega}$, where availability α is defined as

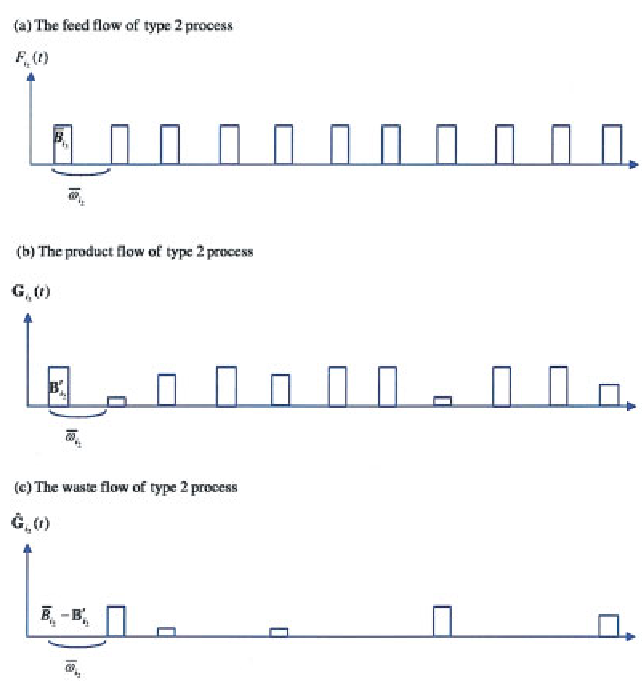

Figure 5. Flow of type 2 process.

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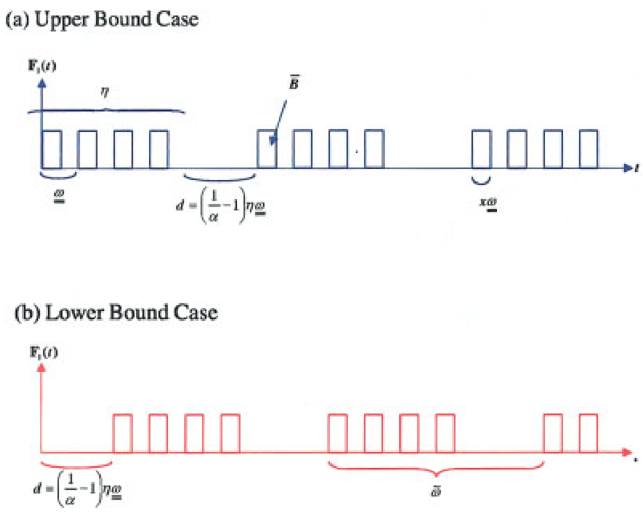

Figure 6. Two extreme cases of random failure.

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$$\alpha \equiv \frac{\text{Actual operating time}}{\text{Actual operating time} + \text{average idle time}} \quad \text{for a type 1 process}$$

$$\left(\alpha_{i_2} \equiv \frac{\text{Average on-spec product quantity}}{\text{Feed quantity}} \quad \text{for a type 2 process} \right) \tag{1}$$

The availability of a production process can be measured by FMEA.[10]

$$\alpha = 1 - \sum_{\substack{all \\ failure \\ modes}} \frac{\sum_{\substack{all \\ repair \\ times}} (Likelihood)(\text{Time to repair})}{\text{Mean time between failures}} \tag{2}$$

The availability of a type 2 process is defined as the ratio of the total successful batch quantity to the total feed batch quantity for a sufficiently large number of batches $\eta_{i_2} = \text{int}[\text{Var}(B'_{i_2})/\delta_1 \varepsilon_1^2] + 1$. In other words, the product batch in Figure 5 will satisfy

$$P\left\{ \left| \frac{1}{\eta_{i_2}} \sum_{l=1}^{\eta_{i_2}} \mathbf{B}'_{i_2(l)} - \alpha_{i_2} \bar{B}_{i_2} \right| < \varepsilon_1 \right\} \geq 1 - \delta_1$$

for a type 2 process. The parameter α_{i_2} can have any real value in the range $0 < \alpha_{i_2} \leq 1$. In the present work, however, α_{i_2} should be chosen such that $\alpha_{i_2}\eta_{i_2}$ is an integer; this is done for the purpose of the graphical analysis used to find the upper and lower bounds of the flow, as shown in Figures 6 and 7. This requirement can be relaxed and a more complicated graphical analysis can be conducted in a future study.

Let us define a long cycle time $\tilde{\omega} \equiv \eta\bar{\omega}$ and a total failure duration in that long cycle time, $d \equiv (1 - \alpha)\tilde{\omega}$. Here $\tilde{\omega}$ has the meaning of the least period within which all random effects diminish with given confidence level and η represents the number of batches in $\tilde{\omega}$. Thus η is the least number of batches within which all random effects diminish with a given confidence level. Note that α and η are given parameters. Both parameters are estimated from the past operating history and/or the characteristics of the same kind of process built in another plant. For type 1 processes, $\underline{\omega}$ is an unknown to be determined. For type 2 processes, $\bar{\omega}_{i_2}$ ($\equiv \underline{\underline{\omega}}_{i_2}$) is an unknown.

From the third type of process prone to random failure, the random characteristics of finished product demand will be defined differently. Both the batch size and cycle time of demand flow are assumed to be random variables. For a given convergence limit $0 < \varepsilon_1 \ll 1$ and confidence levels $0 < \delta_1, \delta_2, \delta_3, \delta_4 \ll 1$, choose the least integer η_m^j for which

$$P\left\{ \left| \frac{1}{\eta_m^j} \sum_{l=1}^{\eta_m^j} \frac{\mathbf{B}_{m(l)}^j}{\boldsymbol{\omega}_{m(l)}^j} - D_m^j \right| < \varepsilon_1 \right\} \geq 1 - \delta_1$$

where D_m^j is average flow rate of the finished product demand. From Tchebycheff's inequality

$$\eta_m^j = \text{int}\left[\frac{\text{Var}(\mathbf{B}_m^j/\boldsymbol{\omega}_m^j)}{\delta_1 \varepsilon_1^2} \right] + 1$$

Then, choose the lowest real number $\tilde{\omega}_m^j$ for which

$$P\left\{ \sum_{l=1}^{\eta_m^j} \boldsymbol{\omega}_{m(l)}^j \geq \tilde{\omega}_m^j \right\} \leq \delta_2$$

and choose the maximum batch size $\bar{\bar{B}}_m^j$ for which $P\{\mathbf{B}_m^j \leq \bar{\bar{B}}_m^j\} \geq 1 - \delta_3$. Then, the minimum number of batches in a long cycle time is $\gamma_m^j \equiv \text{int}[D_m^j \tilde{\omega}_m^j / \bar{\bar{B}}_m^j] + 1$ and adjust the long cycle time of customer demand $\tilde{\omega}_m^j = \gamma_m^j \bar{\bar{B}}_m^j / D_m^j$. Choose the minimum order processing time $\underline{\underline{\omega}}_m^j$ for which $P\{\boldsymbol{\omega}_{m(l)}^j \geqq \underline{\underline{\omega}}_m^j\} \geq 1 -$

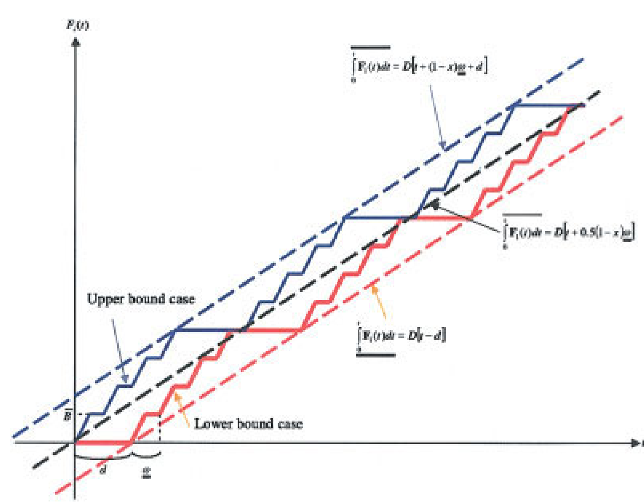

Figure 7. Cumulative flow functions for two extreme cases.

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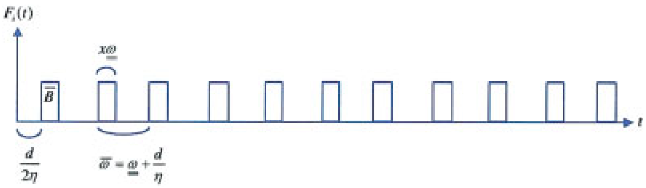

Figure 8. Most probable flow.

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δ_4. Then, the availability of demand flow is $\alpha^j_m \equiv \gamma^j_m \underline{\underline{\omega}}^j_m / \tilde{\omega}^j_m$ and the maximum delay between orders is $d^j_m \equiv (1 - \alpha^j_m)\tilde{\omega}^j_m$. Note that η^j_m, $\tilde{\omega}^j_m$, $\bar{\bar{B}}^j_m$, γ^j_m, $\underline{\underline{\omega}}^j_m$, α^j_m, and d^j_m are either given parameters for demand flow or they can be calculated using the above relationships if the probability distributions of the batch size and cycle time are given.

There exists the following timing relationship between the start-up time of the feedstock streams and the start-up time of the product or waste streams of production processes:

$$t'_i = {}^{\backprime}t_i + \Delta t_i(\cdot) \tag{3}$$

where $\Delta t_i(\cdot)$ is a function of arbitrary variables. Note that $D_{i_1} = \alpha_{i_1}\bar{B}_{i_1}/\underline{\underline{\omega}}_{1_1}$ and $D_{i_2} = \bar{B}_{i_2}/\underline{\underline{\omega}}_{1_2}$ are the average material flow rates through processes i_1 and i_2, respectively. The average material flows of raw material purchasing and finished product demand are respectively denoted by

$$D^j_k = \frac{\alpha^j_k \bar{B}^j_k}{\underline{\underline{\omega}}^j_k} \qquad \text{and} \qquad D^j_m = \frac{\alpha^j_m \bar{\bar{B}}^j_m}{\underline{\underline{\omega}}^j_m} = \frac{\gamma^j_m \bar{\bar{B}}^j_m}{\tilde{\omega}^j_m}$$

From now on, we will use the average flow rates instead of batch sizes in almost all equations. The overall material balance around a storage unit results in the following relationships:

$$\sum_{i_1=1}^{|I_1|} g^j_{i_1} D_{i_1} + \sum_{i_2=1}^{|I_2|} \left(\alpha_{i_2} g^j_{i_2} + \hat{g}^j_{i_2}(1 - \alpha_{i_2})\right) D_{i_2} + \sum_{k=1}^{|K(j)|} D^j_k$$
$$= \sum_{i_1=1}^{|I_1|} f^j_{i_1} D_{i_1} + \sum_{i_2=1}^{|I_2|} f^j_{i_2} D_{i_2} + \sum_{m=1}^{|M(j)|} D^j_m + \sum_{n=1}^{|N(j)|} D^j_n \tag{4}$$

The random inventory holdup level of storage unit j at time t is denoted by $\mathbf{V^j}(t)$ and the initial inventory of this unit is $V^j(0)$. In the present work we need the maximum, minimum, and average inventory levels rather than the actual inventory level. The maximum inventory level will be used to compute the storage size; the minimum inventory level will be used in the optimization constraint that ensures the inventory level is always nonnegative; and the average inventory level will be used to compute the inventory holding cost of the optimization problem. Instead of computing the maximum, minimum, and average inventory levels, the upper/lower bound and average of the accumulated material flow for each stream connected to the storage unit will be computed. Let us denote $\mathbf{F}_1(t)$ as the random flow of a type 1 process. We note that the flow has a constant average flow rate D measured during a long cycle time. This means that in spite of the randomness, the total quantity processed during a long cycle time is constant. $\mathbf{F}_1(t)$ has two extreme flow cases, as shown in Figure 6. The upper bound of the accumulated flow $\int_0^t \mathbf{F}_1(t)dt$ corresponds to the case where all the operating time failures occur at the end of repeated long cycle times. The lower bound of the accumulated flow corresponds to the case where all the operating time failures occur at the start of repeated long cycle times. Note that these two extreme cases can also be defined using probabilistic distributions and confidence levels. Figure 7 shows the corresponding accumulated flow patterns. We can easily find the (linear) upper and lower bounds of $\int_0^t \mathbf{F}_1(t)dt$ from Figure 7.

$$D[t + (1 - x)\underline{\underline{\omega}} + d] \geq \int_0^t \mathbf{F}_1(t)dt \geq D[t - d] \tag{5}$$

Note that $d = [(1/\alpha) - 1]\eta\underline{\underline{\omega}}$ for type 1 processes. The average accumulated flow $\overline{\int_0^t \mathbf{F}_1(t)dt}$ is simply chosen to be equidistant from the upper and lower bounds.

$$\overline{\int_0^t \mathbf{F}_1(t)dt} = D[t + 0.5(1 - x)\underline{\underline{\omega}}] \tag{6}$$

This can be justified from the fact that $\overline{\int_0^t \mathbf{F}_1(t)dt}$ is also the average of the periodic flow with the period of $\bar{\omega} = \underline{\underline{\omega}} + (d/\eta)$, as can be seen in Figures 8 and 9. Note that, for type 1 production processes $x \rightarrow {}^{\backprime}x_{i_1}$, $D \rightarrow f^j_{i_1}D_{i_1}$, and $t \rightarrow t - {}^{\backprime}t_{i_1}$ for feed flow or $x \rightarrow x'_{i_1}$, $D \rightarrow g^j_{i_1}D_{i_1}$, and $t \rightarrow t - {}^{\backprime}t_{i_1} - \Delta t_{i_1}$ for product flow in Eqs. 5 and 6.

Let us denote $F^j_{i_2}(t)$, $\mathbf{G}^j_{i_2}(t)$, and $\hat{\mathbf{G}}^j_{i_2}(t)$ as the feed, product, and waste flows of a type 2 process, as shown in Figures 4 and 5. The analysis to find the upper and lower bounds of $\mathbf{G}^j_{i_2}(t)$ and $\hat{\mathbf{G}}^j_{i_2}(t)$ is the same as that described above for $\mathbf{F}_1(t)$; therefore, Eqs. 5 and 6 can be used with different denotations. $t \rightarrow t - {}^{\backprime}t_{i_2} - \Delta t_{i_2}$, $x \rightarrow x'_{i_2}$, $\underline{\underline{\omega}} \rightarrow \bar{\omega}_{i_2}$, $d \rightarrow d_{i_2}$, and $D \rightarrow g^j_{i_2}\alpha_{i_2}D_{i_2}$ will be applied to product flow $\mathbf{G}^j_{i_2}(t)$; and $t \rightarrow t - {}^{\backprime}t_{i_2} - \Delta t_{i_2}$, $x \rightarrow$

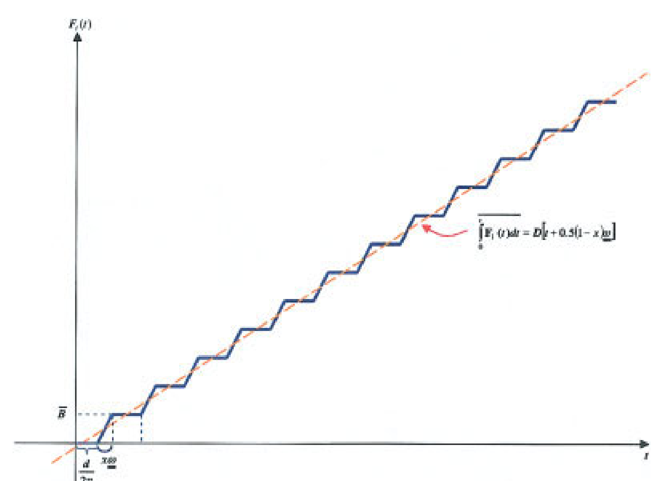

Figure 9. Average flow of the most probable flow.

[Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com]

$\alpha_{i_2}x'_{i_2}$, $\underline{\underline{\omega}} \to \bar{\omega}_{i_2}$, $d \to \hat{d}_{i_2}$, and $D \to \hat{g}^j_{i_2}(1-\alpha_{i_2})D_{i_2}$ will be applied to waste flow $\hat{\mathbf{G}}^j_{i_2}(t)$.

$$g^j_{i_2}\alpha_{i_2}D_{i_2}[t - {}^{\backprime}t_{i_2} - \Delta t_{i_2} + (1-x'_{i_2})\bar{\omega}_{i_2} + d_{i_2}] \geq \int_0^t \mathbf{G}^j_{i_2}(t)dt \geq g^j_{i_2}\alpha_{i_2}D_{i_2}[t - {}^{\backprime}t_{i_2} - \Delta t_{i_2} - d_{i_2}] \quad (7)$$

$$\hat{g}^j_{i_2}(1-\alpha_{i_2})D_{i_2}[t - {}^{\backprime}t_{i_2} - \Delta t_{i_2} + (1-\alpha_{i_2}x'_{i_2})\bar{\omega}_{i_2} + \hat{d}_{i_2}] \geq \int_0^t \hat{\mathbf{G}}^j_{i_2}(t)dt \geq \hat{g}^j_{i_2}(1-\alpha_{i_2})D_{i_2}[t - {}^{\backprime}t_{i_2} - \Delta t_{i_2} - \hat{d}_{i_2}] \quad (8)$$

Note that $d_{i_2} = (1-\alpha_{i_2})\eta_{i_2}\bar{\omega}_{i_2}$ for product flow and $\hat{d}_{i_2} = \alpha_{i_2}\eta_{i_2}\bar{\omega}_{i_2}$ for waste flow. The averages of $\int_0^t \mathbf{G}^j_{i_2}(t)dt$ and $\int_0^t \hat{\mathbf{G}}^j_{i_2}(t)dt$ are also chosen to be equidistant from the lower and upper bounds in Eqs. 7 and 8, respectively.

$$\overline{\int_0^t \mathbf{G}^j_{i_2}(t)dt} = g^j_{i_2}\alpha_{i_2}D_{i_2}[t - {}^{\backprime}t_{i_2} - \Delta t_{i_2} + 0.5(1-x'_{i_2})\bar{\omega}_{i_2}] \quad (9)$$

$$\overline{\int_0^t \hat{\mathbf{G}}^j_{i_2}(t)dt} = \hat{g}^j_{i_2}(1-\alpha_{i_2}) \times D_{i_2}[t - {}^{\backprime}t_{i_2} - \Delta t_{i_2} + 0.5(1-\alpha_{i_2}x'_{i_2})\bar{\omega}_{i_2}] \quad (10)$$

The feed flow of a type 2 process, $F^j_{i_2}(t)$, does not have any failures. The analysis to find the upper and lower bounds of $\int_0^t F^j_{i_2}(t)dt$ follows that presented in our previous study.[13]

$$f^j_{i_2}D_{i_2}[t - {}^{\backprime}t_{i_2} + (1 - {}^{\backprime}x_{i_2})\bar{\omega}_{i_2}] \geq \int_0^t F^j_{i_2}(t)dt \geq f^j_{i_2}D_{i_2}[t - {}^{\backprime}t_{i_2}] \quad (11)$$

Note that the average flow rate of $F^j_{i_2}(t)$ is $f^j_{i_2}D_{i_2}$. The average of $\int_0^t F^j_{i_2}(t)dt$ is selected as the line equidistant from the upper and lower bounds.[13]

$$\overline{\int_0^t F^j_{i_2}(t)dt} = f^j_{i_2}D_{i_2}[t - {}^{\backprime}t_{i_2} + 0.5(1 - {}^{\backprime}x_{i_2})\bar{\omega}_{i_2}] \quad (12)$$

The upper/lower bound and average of the finished product demand flow and waste disposal flow follow the patterns of $\mathbf{F}_1(t)$, which are shown in Figures 6 and 7. It should be noted that there can be other choices of average flows, depending on the characteristics of the random variations in the specific system being analyzed.

The upper bound of the inventory level $\overline{\overline{V^j}}$ is computed by adding the upper bounds of all incoming flows and subtracting the lower bounds of all outgoing flows from the initial inventory. Incoming flows are raw material purchase, product flows from production processes, and waste flows from production processes. Outgoing flows are feed flows to production processes, finished product demand, and waste disposal sinks.

$$\begin{aligned}
\overline{\overline{V^j}} = V^j(0) &- \sum_{k=1}^{|K(j)|} D^j_k\left[t^j_k - (1-x^j_k)\underline{\underline{\omega}}^j_k - \left(\frac{1}{\alpha^j_k} - 1\right)\eta^j_k\underline{\underline{\omega}}^j_k\right] \\
&- \sum_{i_2=1}^{|I_2|} g^j_{i_2}\alpha_{i_2}D_{i_2}[{}^{\backprime}t_{i_2} + \Delta t_{i_2} - ((1-x'_{i_2}) + (1-\alpha_{i_2})\eta_{i_2})\bar{\omega}_{i_2}] \\
&- \sum_{i_2=1}^{|I_2|} \hat{g}_{i_2}(1-\alpha_{i_2})D_{i_2}[{}^{\backprime}t_{i_2} + \Delta t_{i_2} - ((1-x'_{i_2}) + \alpha_{i_2}\eta_{i_2})\bar{\omega}_{i_2}] \\
&- \sum_{i_1=1}^{|I_1|} g^j_{i_1}D_{i_1}\left[{}^{\backprime}t_{i_1} + \Delta t_{i_1} - (1-x'_{i_1})\underline{\underline{\omega}}_{i_1} - \left(\frac{1}{\alpha_{i_1}} - 1\right)\eta_{i_1}\underline{\underline{\omega}}_{i_1}\right] \\
&+ \sum_{i_1=1}^{|I_1|} f^{[\tau]}_{i_1}D^{[\tau]}_{i_1}\left[{}^{\backprime}t^{[\tau]}_{i_1} + \left(\frac{1}{\alpha_{i_1}} - 1\right)\eta_{i_1}\underline{\underline{\omega}}_{i_1}\right] + \sum_{i_2=1}^{|I_2|} f_{i_2}D_{i_2}[{}^{\backprime}t_{i_2}] \\
&+ \sum_{m=1}^{|M(j)|} D^j_m[t^j_m + d^j_m] + \sum_{n=1}^{|N(j)|} D^j_n\left[t^j_n + \left(\frac{1}{\alpha^j_n} - 1\right)\eta^j_n\underline{\underline{\omega}}^j_n\right] \quad (13)
\end{aligned}$$

The lower bound of the inventory level $\underline{\underline{V^j}}$ is computed by adding the lower bounds of all incoming flows and subtracting the upper bounds of all outgoing flows from the initial inventory:

$$\begin{aligned}
\underline{\underline{V^j}} = V^j(0) &- \sum_{k=1}^{|K(j)|} D^j_k\left[t^j_k + \left(\frac{1}{\alpha^j_k} - 1\right)\eta^j_k\underline{\underline{\omega}}^j_k\right] \\
&- \sum_{i_1=1}^{|I_1|} g^j_{i_1}D_{i_1}\left[{}^{\backprime}t_{i_1} + \Delta t_{i_1} + \left(\frac{1}{\alpha_{i_1}} - 1\right)\eta_{i_1}\underline{\underline{\omega}}_{i_1}\right] \\
&+ \sum_{i_1=1}^{|I_1|} f^j_{i_1}D_{i_1}\left[{}^{\backprime}t_{i_1} - (1 - {}^{\backprime}x_{i_1})\underline{\underline{\omega}}_{i_1} - \left(\frac{1}{\alpha_{i_1}} - 1\right)\eta_{i_1}\underline{\underline{\omega}}_{i_1}\right] \\
&- \sum_{i_2=1}^{|I_2|} g^j_{i_2}\alpha_{i_2}D_{i_2}[{}^{\backprime}t_{i_2} + \Delta t_{i_2} + (1-\alpha_{i_2})\eta_{i_2}\bar{\omega}_{i_2}] \\
&- \sum_{i_2=1}^{|I_2|} \hat{g}^j_{i_2}(1-\alpha_{i_2})D_{i_2}[{}^{\backprime}t_{i_2} + \Delta t_{i_2} + \alpha_{i_2}\eta_{i_2}\bar{\omega}_{i_2}] \\
&+ \sum_{i_2=1}^{|I_2|} f_{i_2}D_{i_2}[{}^{\backprime}t_{i_2} - (1 - {}^{\backprime}x_{i_2})\bar{\omega}_{i_2}] \\
&+ \sum_{m=1}^{|M(j)|} D^j_m[t^j_m - (1-x^j_m)\underline{\underline{\omega}}^j_m - d^j_m] \\
&+ \sum_{n=1}^{|N(j)|} D^j_n\left[t^j_n - (1-x^j_n)\underline{\underline{\omega}}^j_n - \left(\frac{1}{\alpha^j_n} - 1\right)\eta^j_n\underline{\underline{\omega}}^j_n\right] \quad (14)
\end{aligned}$$

The average inventory level $\overline{V^j}$ is computed by summing the averages of the accumulated flows.

$$\overline{V^j} = V^j(0) + \sum_{k=1}^{|K(j)|} D_k^j[0.5(1-x_k^j)\underline{\underline{\omega}}_k^j - t_k^j]$$
$$+ \sum_{i_1=1}^{|I_1|} g_{i_1}^j D_{i_1}[0.5(1-x'_{i_1})\underline{\underline{\omega}}_{i_1} - {}'t_{i_1} - \Delta t_{i_1}]$$
$$+ \sum_{i_2=1}^{|I_2|} g_{i_2}^j \alpha_{i_2} D_{i_2}[0.5(1-x'_{i_2})\bar{\omega}_{i_2} - {}'t_{i_2} - \Delta t_{i_2}]$$
$$+ \sum_{i_2=1}^{|I_2|} \hat{g}_{i_2}^j (1-\alpha_{i_2}) D_{i_2}[0.5(1-x'_{i_2})\bar{\omega}_{i_2} - {}'t_{i_2} - \Delta t_{i_2}]$$
$$- \sum_{i_1=1}^{|I_1|} f_{i_1}^j D_{i_1}[0.5(1-{}'x_{i_1})\underline{\underline{\omega}}_{i_1} - {}'t_{i_1}]$$
$$- \sum_{i_2=1}^{|I_2|} f_{i_2}^j D_{i_2}[0.5(1-{}'x_{i_2})\bar{\omega}_{i_2} - {}'t_{i_2}]$$
$$- \sum_{m=1}^{|M(j)|} D_m^j[0.5(1-x_m^j)\underline{\underline{\omega}}_m^j - t_m^j]$$
$$- \sum_{N=1}^{|N(j)|} D_n^j[0.5(1-x_n^j)\underline{\underline{\omega}}_n^j - t_n^j] \quad (15)$$

The purchasing setup cost of raw material j is denoted by A_k^j (\$/order) and the setup cost of process i is denoted by A_i (\$/batch). Note that these setup costs include costs associated with shutdown or failure maintenance. The annual inventory holding cost of storage unit j is denoted by H^j (\$/L/year). We assume that the capital cost is proportional to the (maximum) process capacity to obtain an analytical solution. Suppose that a_k^j (\$/year/L) is the annual capital cost of the purchasing facility for raw material j, a_i (\$/year/L) is the annual capital cost of process i, and b^j (\$/year/L) is the annual capital cost of storage unit j. The purchase price of raw material j from supplier k is denoted P_k^j (\$/L), and the sales price of finished product j to consumer m is P_m^j(\$/L). We use the same parameter definition scheme for the parameters associated with waste disposal flow, that is, A_n^j (\$/batch), a_n^j (\$/year/L), and P_n^j (\$/L). The objective function for the design of the batch-storage network is to minimize the annualized expectation of total cost, which consists of the raw material procurement cost, the setup cost of processes, the waste disposal cost, the inventory holding cost of storage units, and the capital cost of the processes and storage units at a given availability and number of batches in a long cycle time of each process.

$$ATC = \sum_{j=1}^{|J|}\sum_{k=1}^{|K(j)|}\left[\frac{\eta_k^j A_k^j}{\tilde{\omega}_k^j} + a_k^j \bar{B}_k^j + P_k^j D_k^j\right]$$
$$+ \sum_{i_1=1}^{|I_1|}\left[\frac{\eta_{i_1} A_{i_1}}{\tilde{\omega}_{i_1}} + a_{i_1}\bar{B}_{i_1}\right] + \sum_{i_2=1}^{|I_2|}\left[\frac{\eta_{i_2} A_{i_2}}{\tilde{\omega}_{i_2}} + a_{i_2}\bar{B}_{i_2}\right]$$
$$+ \sum_{j=1}^{|J|}\left[H^j \overline{V^j} + b^j \overline{\overline{V}}{}^j\right] - \sum_{j=1}^{|J|}\sum_{M=1}^{|M(j)|} P_m^j D_m^j$$
$$+ \sum_{j=1}^{|J|}\sum_{n=1}^{|N(j)|}\left[\frac{\eta_n^j A_n^j}{\tilde{\omega}_n^j} + a_n^j \bar{B}_n^j + P_n^j D_n^j\right] \quad (16)$$

The average flow rates of demand flow D_m^jcan be considered as either an unknown or a parameter depending on the problem. Remember that we have η setups within $\tilde{\omega}$ for type 1 processes and η_{i_2} setups within $\tilde{\omega}_{i_2}$ for type 2 processes. We choose the design variables of optimization as the minimum cycle times $\underline{\underline{\omega}}$, that is, $\underline{\underline{\omega}}_k^j$, $\underline{\underline{\omega}}_{i_1}$, $\underline{\underline{\omega}}_{i_2}$ $(=\bar{\omega}_{i_2})$, and $\underline{\underline{\omega}}_n^j$. From the definitions of variables and parameters, $\eta A/\tilde{\omega} = \alpha A/\underline{\underline{\omega}}$ and $\bar{B} = D\underline{\underline{\omega}}/\alpha$ for type 1 processes and $\eta_{i_2} A_{i_2}/\tilde{\omega}_{i_2} = A_{i_2}/\bar{\omega}_{i_2}$ and $\bar{B}_{i_2} = D_{i_2}\bar{\omega}_{i_2}/\alpha_{i_2}$ for type 2 processes. The $\overline{V^j}$ and $\overline{\overline{V}}{}^j$ in Eq. 16 are further developed from Eqs. 13 and 15. The optimization constraints are no depletion of all storage units, $0 \le \underline{\underline{V}}^j$, where $\underline{\underline{V}}^j$ is given in Eq. 14.

$$V^j(0) - \sum_{k=1}^{|K(j)|} D_k^j\left[t_k^j + \left(\frac{1}{\alpha_k^j} - 1\right)\eta_k^j \underline{\underline{\omega}}_k^j\right] - \sum_{i_1=1}^{|I_1|} g_{i_1}^j D_{i_1}\left[{}'t_{i_1} + \Delta t_{i_1}\right.$$
$$\left. + \left(\frac{1}{\alpha_{i_1}} - 1\right)\eta_{i_1}\underline{\underline{\omega}}_{i_1}\right] + \sum_{i_1=1}^{|I_1|} f_{i_1}^j D_{i_1}\left[{}'t_{i_1} - (1-{}'x_{i_1})\underline{\underline{\omega}}_{i_1}\right.$$
$$\left. - \left(\frac{1}{\alpha_{i_1}} - 1\right)\eta_{i_1}\underline{\underline{\omega}}_{i_1}\right] - \sum_{i_2=1}^{|I_2|} g_{i_2}^j \alpha_{i_2} D_{i_2}[{}'t_{i_2} + \Delta t_{i_2} + (1$$
$$- \alpha_{i_2})\eta_{i_2}\bar{\omega}_{i_2}] - \sum_{i_2=1}^{|I_2|} \hat{g}_{i_2}^j (1-\alpha_{i_2}) D_{i_2}[{}'t_{i_2} + \Delta t_{i_2} + \alpha_{i_2}\eta_{i_2}\bar{\omega}_{i_2}]$$
$$+ \sum_{i_2=1}^{|I_2|} f_{i_2} D_{i_2}[{}'t_{i_2} - (1-{}'x_{i_2})\bar{\omega}_{i_2}] + \sum_{m=1}^{|M(j)|} D_m^j[t_m^j - (1-x_m^j)\underline{\underline{\omega}}_m^j$$
$$- d_m^j] + \sum_{n=1}^{|N(j)|} D_n^j\left[t_n^j - (1-x_n^j)\underline{\underline{\omega}}_n^j - \left(\frac{1}{\alpha_n^j} - 1\right)\eta_n^j \underline{\underline{\omega}}_n^j\right] \ge 0$$
$$(17)$$

Problem Definition for Multiperiod Problem

The above single period problem assumes constant average flow rates and constant parameters, assumptions that are unrealistic in the real world. Average flow rates and parameters usually vary slowly over time, a property that cannot be reflected in the short-term random variations defined in the previous section. To accommodate long-term variations of all parameters and variables as well as the average flow rates, we now introduce a multiperiod formulation. We have T time periods from $\tau = 0$ to $\tau = T - 1$ with time intervals $\nabla t^{[\tau]}$. It is not necessary for all $\nabla t^{[\tau]}$ to be equal with respect to τ; $\nabla t^{[\tau]}$ should be sufficiently greater than $\tilde{\omega}^{[\tau]}$ but it should take on the lowest values satisfying $\nabla t^{[\tau]} \gg \tilde{\omega}^{[\tau]}$ so that the expressions for the upper and lower bounds (Eqs. 5, 7, and 8) are satisfactory. Hereinafter, we assume that all the parameters and variables in the above section have superscript $[\tau]$, such as $\underline{\underline{\omega}}_k^{j[\tau]}$, $\underline{\underline{\omega}}_{i_1}^{[\tau]}$, $\underline{\underline{\omega}}_{i_2}^{[\tau]}$ $(=\bar{\omega}_{i_2}^{[\tau]})$, and $\underline{\underline{\omega}}_n^{j[\tau]}$. The material balance around a storage unit, Eq. 4, is no longer valid because it is derived based on the assumption of no inventory imbalance within a period. The inventory imbalance of one period can be transferred to the next period. Inventory overstocking commonly has the benefits of allowing the operator to take advantage of a favorable raw material price when price rises are expected or to secure quantity discounts. When product demand varies with time, it may be more economical to increase the inventory during periods of low demand to ease the strain on the production facilities during peak demand periods; this is the seasonal inventory approach. The extent to which this policy should be used is determined by balancing the cost of carrying the seasonal inventories against the cost of changing the production rate and of not fully meeting demand. The inventory variables $\mathbf{V^j}(t)$, $V^j(0)$, $\overline{\overline{V}}^j$, $\underline{\underline{V}}^j$, and $\overline{V^j}$ defined above for the single period problem cannot be used to implement a seasonal inventory. Here we define the multiperiod inventory level at time point $t = 0$ of period τ as $v^{j[\tau]}(0)$, where the actual inventory level is $v^{j[\tau]}(0) + V^{j[\tau]}(0)$. Note that the time point $t = 0$ of period τ is the starting moment of that time period, where $0 \le t \le \nabla t^{[\tau]}$. The material imbalance rate of a storage unit j in time period τ is denoted as $\dot{v}^{j[\tau]}$. Then, Eq. 4 is changed into

$$\dot{v}^{j[\tau]} = \sum_{i_1=1}^{|I_1|} g_{i_1}^{j[\tau]} D_{i_1}^{[\tau]} + \sum_{i_2=1}^{|I_2|} \left(g_{i_2}^{j[\tau]} \alpha_{i_2}^{[\tau]} + \hat{g}_{i_2}^{j[\tau]} (1 - \alpha_{i_2}^{[\tau]}) \right) D_{i_2}^{[\tau]} + \sum_{k=1}^{|K(j)|} D_k^{j[\tau]} - \sum_{i_1=1}^{|I_1|} f_{i_1}^{j[\tau]} D_{i_1}^{[\tau]} - \sum_{i_2=1}^{|I_2|} f_{i_2}^{j[\tau]} D_{i_2}^{[\tau]} - \sum_{m=1}^{|M(j)|} D_m^{j[\tau]} - \sum_{n=1}^{|N(j)|} D_n^{j[\tau]} \tag{18}$$

Because all parameters and variables in one period are constant, $\dot{v}^{j\,[\tau]}$ is a constant value. Therefore, the following inventory replenishment relationship holds:

$$v^{j[\tau+1]}(0) = v^{j[\tau]}(0) + \dot{v}^{j[\tau]} \nabla t^{[\tau]},\ v^{j[0]}(0) = v_0^j,\ \underline{\underline{v_T^j}} \le v^{j[T]}(0) \le \overline{\overline{v_T^j}} \tag{19}$$

Let us denote the upper and lower bounds and average of the multiperiod inventory level in time period τ as $\overline{\overline{v^{j[\tau]}}}$, $\underline{\underline{v^{j[\tau]}}}$, and $\overline{v^{j[\tau]}}$, respectively. Then,

$$\overline{\overline{v^{j[\tau]}}} = \max\{v^{j[\tau]}(0), v^{j[\tau+1]}(0)\} + \overline{\overline{V^{j[\tau]}}} \tag{20}$$

$$\underline{\underline{v^{j[\tau]}}} = \min\{v^{j[\tau]}(0), v^{j[\tau+1]}(0)\} + \underline{\underline{V^{j[\tau]}}} \tag{21}$$

$$\overline{v^{j[\tau]}} = 0.5(v^{j[\tau]}(0) + v^{j[\tau+1]}(0)) + \overline{V^{j[\tau]}} \tag{22}$$

The objective function of the single period problem, Eq. 16, is denoted as $ATC^{[\tau]}$ from now on. The terms $\overline{\overline{V^j}}$ and $\overline{V^j}$ in $ATC^{[\tau]}$ are replaced with $\overline{\overline{v^{j[\tau]}}}$ and $\overline{v^{j[\tau]}}$. Then, the multiperiod objective function is the sum of the discounted single period objective minus the salvage value of the last inventory:

$$TC = \sum_{\tau=0}^{T-1} ATC^{[\tau]} \nabla t^{[\tau]} \prod_{\tau'=0}^{\tau-1} (1 + \rho^{[\tau']})^{-1} - \sum_{j=1}^{|J|} h^{j[T]} v^{j[T]}(0) \tag{23}$$

where $\rho^{[\tau]}$ (\$/\$) is the discount rate of period τ used to compute net present value and $h^{j[\tau]}$ (\$/L) is the salvage value of material j at the end of period T. The constraints of the multiperiod problem are

$$\underline{\underline{v^{j[\tau]}}} \ge \min\{v^{j[\tau]}(0), v^{j[\tau+1]}(0)\} \tag{24}$$

Note that Eq. 24 returns to Eq. 17. Table 1 summarizes the multiperiod optimization problem.

Kuhn–Tucker Solutions of the Multiperiod Problem

The multiperiod problem has design variables $\omega_k^{j[\tau]}$, $\omega_i^{[\tau]}$, $\omega_n^{j[\tau]}$, $t_k^{j[\tau]}$, ${}^{\backprime}t_i^{[\tau]}$, and $t_n^{j[\tau]}$, with the objective function given in Eq. 23 and constraints given in Eq. 24. The solution procedure to solve the Kuhn–Tucker conditions is given in Appendix A. Optimal minimum cycle times and batch sizes are

$$*\underline{\underline{\omega}}_k^{j[\tau]} = \sqrt{\frac{A_k^{j[\tau]}}{D_k^{j[\tau]} \Psi_k^{j[\tau]}}},\quad *\bar{B}_k^{j[\tau]} = \frac{1}{\alpha_k^{j[\tau]}} \sqrt{\frac{A_k^{j[\tau]} D_k^{j[\tau]}}{\Psi_k^{j[\tau]}}} \tag{25}$$

where

$$\Psi_k^{j[\tau]} = \frac{1}{\alpha_k^{j[\tau]}} \left[(0.5H^{j[\tau]} + b^{j[\tau]}) \times \left\{ 1 - x_k^{j[\tau]} + 2\left(\frac{1}{\alpha_k^{j[\tau]}} - 1 \right) \eta_k^{j[\tau]} \right\} + \frac{a_k^{j[\tau]}}{\alpha_k^{j[\tau]}} \right] \tag{26}$$

$$*\underline{\underline{\omega}}_n^{j[\tau]} = \sqrt{\frac{A_n^{j[\tau]}}{D_n^{j[\tau]} \Psi_n^{j[\tau]}}},\quad *\bar{B}_n^{j[\tau]} = \frac{1}{\alpha_n^{j[\tau]}} \sqrt{\frac{A_n^{j[\tau]} D_n^{j[\tau]}}{\Psi_n^{j[\tau]}}} \tag{27}$$

where

$$\Psi_n^{j[\tau]} = \frac{1}{\alpha_n^{j[\tau]}} \left[(0.5H^{j[\tau]} + b^{j[\tau]}) \times \left\{ 1 - x_n^{j[\tau]} + 2\left(\frac{1}{\alpha_n^{j[\tau]}} - 1 \right) \eta_n^{j[\tau]} \right\} + \frac{a_n^{j[\tau]}}{\alpha_n^{j[\tau]}} \right] \tag{28}$$

$$*\underline{\underline{\omega}}_{i_1}^{[\tau]} = \sqrt{\frac{A_{i_1}^{[\tau]}}{\Psi_{i_1}^{[\tau]} D_{i_1}^{[\tau]}}},\quad *\bar{B}_{i_1}^{[\tau]} = \frac{1}{\alpha_{i_1}^{[\tau]}} \sqrt{\frac{A_{i_1}^{[\tau]} D_{i_1}^{[\tau]}}{\Psi_{i_1}^{[\tau]}}} \tag{29}$$

where

Table 1. Multiperiod Problem

$$\text{Minimize } TC = \sum_{\tau=0}^{T-1} ATC^{[\tau]} \nabla t^{[\tau]} \prod_{\tau'=0}^{\tau-1} (1 + P^{[\tau']})^{-1} - \sum_{j=1}^{|J|} h^{j[T]} \mathcal{v}^{j[T]}(0)$$

with respect to $\underline{\underline{\omega}}_k^{j[\tau]},\ \underline{\underline{\omega}}_{i_1}^{[\tau]},\ \bar{\omega}_{i_2}^{[\tau]},\ \underline{\underline{\omega}}_n^{j[\tau]}$ and $t_k^{j[\tau]},\ {}^{\backprime}t_{i_1}^{[\tau]},\ {}^{\backprime}t_{i_2}^{[\tau]},\ t_n^{j[\tau]}$
such that $\underline{\underline{\mathcal{v}^{j[\tau]}}} \geq \min\{\overline{\mathcal{v}^{j[\tau]}(0)},\ \overline{\mathcal{v}^{j[\tau+1]}}(0)\}$
where

$$ATC^{[\tau]} = \sum_{j=1}^{|J|} \sum_{k=1}^{|K(j)|} \left[\frac{\alpha_k^{j(\tau)} A_k^{j(\tau)}}{\underline{\underline{\omega}}_k^{j[\tau]}} + a_k^{j[\tau]} \left(\frac{D_k^{j[\tau]} \underline{\underline{\omega}}_k^{j[\tau]}}{\alpha_k^{j[\tau]}} \right) + P_k^{j[\tau]} D_k^{j[\tau]} \right] + \sum_{i_1=1}^{|I_1|} \left[\frac{\alpha_{i_1}^{[\tau]} A_{i_1}^{[\tau]}}{\underline{\underline{\omega}}_{i_1}^{[\tau]}} + a_{i_1}^{[\tau]} \left(\frac{D_{i_1}^{[\tau]} \underline{\underline{\omega}}_{i_1}^{[\tau]}}{\alpha_{i_1}^{[\tau]}} \right) \right] + \sum_{i_2=1}^{|I_2|} \left[\frac{A_{i_2}^{[\tau]}}{\bar{\omega}_{i_2}^{[\tau]}} + a_{i_2}^{[\tau]} \left(\frac{D_{i_2}^{[\tau]} \bar{\omega}_{i_2}^{[\tau]}}{\alpha_{i_2}^{[\tau]}} \right) \right] + \sum_{j=1}^{|J|} [H^{j[\tau]} \overline{\mathcal{v}^{j[\tau]}} + b^{j[\tau]} \overline{\overline{\mathcal{v}^{j[\tau]}}}]$$
$$- \sum_{j=1}^{|J|} \sum_{M=1}^{|M(j)|} P_m^{j[\tau]} D_m^{j[\tau]} + \sum_{j=1}^{|J|} \sum_{n=1}^{|N(j)|} \left[\frac{\alpha_n^{j[\tau]} A_n^{j[\tau]}}{\underline{\underline{\omega}}_n^{j[\tau]}} + a_n^{j[\tau]} \left(\frac{D_n^{j[\tau]} \underline{\underline{\omega}}_n^{j[\tau]}}{\alpha_n^{j[\tau]}} \right) + P_n^{j[\tau]} D_n^{j[\tau]} \right]$$

$$\overline{\overline{\mathcal{v}^{j[\tau]}}} = \max\{\mathcal{v}^{j(\tau)}(0),\ \mathcal{v}^{j(\tau+1)}(0)\} + V^{j(\tau)}(0) - \sum_{k=1}^{|K(j)|} D_k^{j[\tau]} \left[t_k^{j[\tau]} - (1 - x_k^{j[\tau]}) \underline{\underline{\omega}}_k^{j[\tau]} - \left(\frac{1}{\alpha_k^{j[\tau]}} - 1 \right) \eta_k^{j[\tau]} \underline{\underline{\omega}}_k^{j[\tau]} \right] - \sum_{i_2=1}^{|I_2|} g_{i_2}^{j[\tau]} \alpha_{i_2}^{[\tau]} D_{i_2}^{[\tau]} [{}^{\backprime}t_{i_2}^{[\tau]} + \Delta t_{i_2}^{[\tau]} - ((1 - x_{i_2}'^{[\tau]})$$
$$+ (1 - \alpha_{i_2}^{[\tau]}) \eta_{i_2}^{[\tau]}) \bar{\omega}_{i_2}^{[\tau]}] - \sum_{i_2=1}^{|I_2|} \hat{g}_{i_2}^{j[\tau]} (1 - \alpha_{i_2}^{[\tau]}) D_{i_2}^{[\tau]} [{}^{\backprime}t_{i_2}^{[\tau]} + \Delta t_{i_2}^{[\tau]} - ((1 - x_{i_2}'^{[\tau]}) + \alpha_{i_2}^{[\tau]} \eta_{i_2}^{[\tau]}) \bar{\omega}_{i_2}^{[\tau]}] - \sum_{i_1=1}^{|I_1|} g_{i_1}^{j[\tau]} D_{i_1}^{[\tau]} \left[{}^{\backprime}t_{i_1}^{[\tau]} + \Delta t_{i_1}^{[\tau]} - (1 - x_{i_1}'^{[\tau]}) \underline{\underline{\omega}}_{i_1}^{[\tau]} - \left(\frac{1}{\alpha_{i_1}^{[\tau]}} - 1 \right) \eta_{i_1}^{[\tau]} \underline{\underline{\omega}}_{i_1}^{[\tau]} \right]$$
$$+ \sum_{i_1=1}^{|I_1|} f_{i_1}^{j[\tau]} D_{i_1}^{[\tau]} \left[{}^{\backprime}t_{i_1}^{[\tau]} + \left(\frac{1}{\alpha_{i_1}^{[\tau]}} - 1 \right) \eta_{i_1}^{[\tau]} \underline{\underline{\omega}}_{i_1}^{[\tau]} \right] + \sum_{i_2=1}^{|I_2|} f_{i_2}^{j[\tau]} D_{i_2}^{[\tau]} [{}^{\backprime}t_{i_2}^{[\tau]}] + \sum_{m=1}^{|M(j)|} D_m^{j[\tau]} [t_m^{j[\tau]} + d_m^{j[\tau]}] + \sum_{n=1}^{|N(j)|} D_n^{j[\tau]} \left[t_n^{j[\tau]} + \left(\frac{1}{\alpha_n^{j[\tau]}} - 1 \right) \eta_n^{j[\tau]} \underline{\underline{\omega}}_n^{j[\tau]} \right]$$

$$\underline{\underline{\mathcal{v}^{j[\tau]}}} = \min\{\mathcal{v}^{j(\tau)}(0),\ \mathcal{v}^{j(\tau+1)}(0)\} + V^{j(\tau)}(0) - \sum_{k=1}^{|K(j)|} D_k^{j[\tau]} \left[t_k^{j[\tau]} + \left(\frac{1}{\alpha_k^{j[\tau]}} - 1 \right) \eta_k^{j[\tau]} \underline{\underline{\omega}}_k^{j[\tau]} \right] - \sum_{i_1=1}^{|I_1|} g_{i_1}^{j[\tau]} D_{i_1}^{[\tau]} \left[{}^{\backprime}t_{i_1}^{[\tau]} + \Delta t_{i_1}^{[\tau]} + \left(\frac{1}{\alpha_{i_1}^{[\tau]}} - 1 \right) \eta_{i_1}^{[\tau]} \underline{\underline{\omega}}_{i_1}^{[\tau]} \right] + \sum_{i_1=1}^{|I_1|} f_{i_1}^{j[\tau]} D_{i_1}^{[\tau]} \left[{}^{\backprime}t_{i_1}^{[\tau]} \right.$$
$$\left. - (1 - {}^{\backprime}x_{i_1}^{[\tau]}) \underline{\underline{\omega}}_{i_1}^{[\tau]} - \left(\frac{1}{\alpha_{i_1}^{[\tau]}} - 1 \right) \eta_{i_1}^{[\tau]} \underline{\underline{\omega}}_{i_1}^{[\tau]} \right] - \sum_{i_2=1}^{|I_2|} g_{i_2}^{j[\tau]} \alpha_{i_2}^{[\tau]} D_{i_2}^{[\tau]} [{}^{\backprime}t_{i_2}^{[\tau]} + \Delta t_{i_2}^{[\tau]} + (1 - \alpha_{i_2}^{[\tau]}) \eta_{i_2}^{[\tau]} \bar{\omega}_{i_2}^{[\tau]}] - \sum_{i_2=1}^{|I_2|} \hat{g}_{i_2}^{j[\tau]} (1 - \alpha_{i_2}^{[\tau]}) D_{i_2}^{[\tau]} [{}^{\backprime}t_{i_2}^{[\tau]} + \Delta t_{i_2}^{[\tau]} + \alpha_{i_2}^{[\tau]} \eta_{i_2}^{[\tau]} \bar{\omega}_{i_2}^{[\tau]}]$$
$$+ \sum_{i_2=1}^{|I_2|} f_{i_2}^{j[\tau]} D_{i_2}^{[\tau]} [{}^{\backprime}t_{i_2}^{[\tau]} - (1 - {}^{\backprime}x_{i_2}^{[\tau]}) \bar{\omega}_{i_2}^{[\tau]}] + \sum_{m=1}^{|M(j)|} D_m^{j[\tau]} [t_m^{j[\tau]} - (1 - x_m^{j[\tau]}) \underline{\underline{\omega}}_m^{j[\tau]} - d_m^{j[\tau]}] + \sum_{n=1}^{|N(j)|} D_n^{j[\tau]} \left[t_n^{j[\tau]} - (1 - x_n^{j[\tau]}) \underline{\underline{\omega}}_n^{j[\tau]} - \left(\frac{1}{\alpha_n^{j[\tau]}} - 1 \right) \eta_n^{j[\tau]} \underline{\underline{\omega}}_n^{j[\tau]} \right]$$

$$\overline{\mathcal{v}^{j[\tau]}} = 0.5(\mathcal{v}^{j[\tau]}(0) + \mathcal{v}^{j[\tau+1]}(0)) + V^{j[\tau]}(0) + \sum_{k=1}^{|K(j)|} D_k^{j[\tau]} [0.5(1 - x_k^{j[\tau]}) \underline{\underline{\omega}}_k^{j[\tau]} - t_k^{j[\tau]}] + \sum_{i_1=1}^{|I_1|} g_{i_1}^{j[\tau]} D_{i_1}^{[\tau]} [0.5(1 - x_{i_1}'^{[\tau]}) \underline{\underline{\omega}}_{i_1}^{[\tau]} - {}^{\backprime}t_{i_1}^{[\tau]} - \Delta t_{i_1}^{[\tau]}]$$
$$+ \sum_{i_2=1}^{|I_2|} g_{i_2}^{j[\tau]} \alpha_{i_2}^{[\tau]} D_{i_2}^{[\tau]} [0.5(1 - x_{i_2}'^{[\tau]}) \bar{\omega}_{i_2}^{[\tau]} - {}^{\backprime}t_{i_2}^{[\tau]} - \Delta t_{i_2}^{[\tau]}] + \sum_{i_2=1}^{|I_2|} \hat{g}_{i_2}^{j[\tau]} (1 - \alpha_{i_2}^{[\tau]}) D_{i_2}^{[\tau]} [0.5(1 - x_{i_2}'^{[\tau]}) \bar{\omega}_{i_2}^{[\tau]} - {}^{\backprime}t_{i_3}^{[\tau]} - \Delta t_{i_3}^{[\tau]}] - \sum_{i_1=1}^{|I_1|} f_{i_1}^{j[\tau]} D_{i_1}^{[\tau]} [0.5(1 - {}^{\backprime}x_{i_1}^{[\tau]}) \underline{\underline{\omega}}_{i_1}^{[\tau]} - {}^{\backprime}t_{i_1}^{[\tau]}]$$
$$- \sum_{i_2=1}^{|I_2|} f_{i_2}^{j[\tau]} D_{i_2}^{[\tau]} [0.5(1 - {}^{\backprime}x_{i_2}^{[\tau]}) \bar{\omega}_{i_2}^{[\tau]} - {}^{\backprime}t_{i_2}^{[\tau]}] - \sum_{m=1}^{|M(j)|} D_m^{j[\tau]} [0.5(1 - x_m^{j[\tau]}) \underline{\underline{\omega}}_m^{j[\tau]} - t_m^{j[\tau]}] - \sum_{N=1}^{|N(j)|} D_n^{j[\tau]} [0.5(1 - x_n^{j[\tau]}) \underline{\underline{\omega}}_n^{j[\tau]} - t_n^{j[\tau]}]$$

$$\Psi_{i_1}^{[\tau]} = \sum_{j=1}^{|J|} \frac{1}{\alpha_{i_1}^{[\tau]}} \left[\frac{a_{i_1}^{[\tau]}}{\alpha_{i_1}^{[\tau]}} + \left(\frac{H^{j[\tau]}}{2} + b^{j[\tau]} \right) \{ (1 - {}^{\backprime}x_{i_1}^{[\tau]}) f_{i_1}^{j[\tau]} + (1 - x_{i_1}'^{[\tau]}) g_{i_1}^{j[\tau]} \} \right] + \sum_{j=1}^{|J|} \frac{2}{\alpha_{i_1}^{[\tau]}} \left[\left(\frac{1}{\alpha_{i_1}^{[\tau]}} - 1 \right) \eta_{i_1}^{[\tau]} (f_{i_1}^{j[\tau]} + g_{i_1}^{j[\tau]}) \right] \times \left(\frac{H^{j[\tau]}}{2} + b^{j[\tau]} \right) \quad (30)$$

$$*\bar{\omega}_{i_2}^{[\tau]} = \sqrt{\frac{A_{i_2}^{[\tau]}}{\Psi_{i_2}^{[\tau]} D_{i_2}^{[\tau]}}} \qquad *\bar{B}_{i_2}^{[\tau]} = \sqrt{\frac{A_{i_2}^{[\tau]} D_{i_2}^{[\tau]}}{\Psi_{i_2}^{[\tau]}}} \quad (31)$$

where

$$\Psi_{i_2}^{[\tau]} = \frac{a_{i_2}^{[\tau]}}{\alpha_{i_2}^{[\tau]}} + (1 - {}^{\backprime}x_{i_2}^{[\tau]}) \sum_{j=1}^{|J|} (0.5 H^{j[\tau]} + b^{j[\tau]}) f_{i_2}^{j[\tau]} + \alpha_{i_2}^{[\tau]} [(1 - x_{i_2}'^{[\tau]}) + 2(1 - \alpha_{i_2}^{[\tau]}) \eta_{i_2}^{[\tau]}] \sum_{j=1}^{|J|} (0.5 H^{j[\tau]} + b^{j[\tau]}) g_{i_2}^{j[\tau]} + (1 - \alpha_{i_2}^{[\tau]}) [(1 - x_{i_2}'^{[\tau]}) + 2 \alpha_{i_2}^{[\tau]} \eta_{i_2}^{[\tau]}] \sum_{j=1}^{|J|} (0.5 H^{j[\tau]} + b^{j[\tau]}) \hat{g}_{i_2}^{j[\tau]} \quad (32)$$

The optimal start-up times are

$$2 \sum_{k=1}^{|K(j)|} D_k^{j[\tau]} t_k^{j[\tau]} + \sum_{i=1}^{|I|} (g_{i_1}^{j[\tau]} - f_{i_1}^{j[\tau]}) D_{i_1}^{[\tau]} {}^{\backprime}t_{i_1}^{[\tau]} + \sum_{i_2=1}^{|I_2|} (g_{i_2}^{j[\tau]} \alpha_{i_2}^{[\tau]} + \hat{g}_{i_2}^{j[\tau]} (1$$

$$- \alpha_{i_2}^{[\tau]}) - f_{i_2}^{j[\tau]}) D_{i_2}^{[\tau]} `t_{i_2}^{[\tau]} - \sum_{n=1}^{|N(j)|} D_n^{j[\tau]} t_n^{j[\tau]} = V^{j[\tau]}(0) - \sum_{k=1}^{|K(j)|} D_k^{j[\tau]}$$

$$\times \left(\frac{1}{\alpha_k^{j[\tau]}} - 1 \right) \eta_k^{j[\tau]} \underline{\underline{\omega}}_k^{j[\tau]} - \sum_{i_1=1}^{|I_1|} g_{i_1}^{j[\tau]} D_{i_1}^{[\tau]} \Delta t_{i_1}^{[\tau]} - \sum_{i_1=1}^{|I_1|} (f_{i_1}^{j[\tau]}$$

$$+ g_{i_1}^{j[\tau]}) D_{i_1}^{[\tau]} \left(\frac{1}{\alpha_{i_1}^{[\tau]}} - 1 \right) \eta_{i_1}^{[\tau]} \underline{\underline{\omega}}_{i_1}^{[\tau]} - \sum_{i_1=1}^{|I_1|} (1 - `x_{i_1}^{[\tau]}) f_{i_1}^{j[\tau]} D_{i_1}^{[\tau]} \underline{\underline{\omega}}_{i_1}^{[\tau]}$$

$$- \sum_{i_2=1}^{|I_2|} (g_{i_2}^{j[\tau]} \alpha_{i_2}^{[\tau]} + \hat{g}_{i_2}^{j[\tau]} (1 - \alpha_{i_2}^{[\tau]})) D_{i_2}^{[\tau]} \Delta t_{i_2}^{[\tau]} - \sum_{i_2=1}^{|I_2|} (1$$

$$- `x_{i_2}^{[\tau]}) f_{i_2}^{j[\tau]} D_{i_2}^{[\tau]} \bar{\omega}_{i_2}^{[\tau]} - \sum_{i_2=1}^{|I_2|} g_{i_2}^{j[\tau]} \alpha_{i_2}^{[\tau]} (1 - \alpha_{i_2}^{[\tau]}) \eta_{i_2}^{[\tau]} D_{i_2}^{[\tau]} \bar{\omega}_{i_2}^{[\tau]}$$

$$- \sum_{i_2=1}^{|I_2|} \hat{g}_{i_2}^{j[\tau]} (1 - \alpha_{i_2}^{[\tau]}) \alpha_{i_2}^{[\tau]} \eta_{i_2}^{[\tau]} D_{i_2}^{[\tau]} \bar{\omega}_{i_2}^{[\tau]} - \sum_{m=1}^{|M(j)|} (1 - x_m^{j[\tau]}) D_m^{j[\tau]} \underline{\underline{\omega}}_m^{j[\tau]}$$

$$+ \sum_{m=1}^{|M(j)|} D_m^{j[\tau]} t_m^{j[\tau]} - \sum_{m=1}^{|M(j)|} D_m^{j[\tau]} d_m^{j[\tau]} - \sum_{n=1}^{|N(j)|} (1 - x_n^{j[\tau]}) D_n^{j[\tau]} \underline{\underline{\omega}}_n^{j[\tau]}$$

$$- \sum_{n=1}^{|N(j)|} D_n^{j[\tau]} \left(\frac{1}{\alpha_n^{j[\tau]}} - 1 \right) \eta_n^{j[\tau]} \underline{\underline{\omega}}_n^{j[\tau]} \qquad \forall\, j, \tau \quad (33)$$

The optimal storage sizes, derived from Eqs. 20 and 33 are

$$\overline{\overline{v^{j[\tau]}}} = \max\{v^{j[\tau]}(0), v^{j[\tau+1]}(0)\} + \sum_{m=1}^{|M(j)|} (1 - x_m^{j[\tau]}) D_m^{j[\tau]} \underline{\underline{\omega}}_m^{j[\tau]}$$

$$+ 2 \sum_{m=1}^{|M(j)|} D_m^{j[\tau]} d_m^{j[\tau]}$$

$$+ \sum_{i_1=1}^{|I_1|} \Bigg[(1 - `x_{i_1}^{[\tau]}) f_{i_1}^{j[\tau]} + (1 - x_{i_1}'^{[\tau]}) g_{i_1}^{j[\tau]} + 2(f_{i_1}^{j[\tau]}$$

$$+ g_{i_1}^{j[\tau]}) \left(\frac{1}{\alpha_{i_1}^{[\tau]}} - 1 \right) \eta_{i_1}^{[\tau]} \Bigg] D_{i_1}^{[\tau]} (* \underline{\underline{\omega}}_{i_1}^{[\tau]})$$

$$+ \sum_{k=1}^{|K(j)|} \left[(1 - x_k^j) + 2 \left(\frac{1}{\alpha_k^{j[\tau]}} - 1 \right) \eta_k^{j[\tau]} \right] D_k^{j[\tau]} (* \underline{\underline{\omega}}_k^{j[\tau]})$$

$$+ \sum_{n=1}^{|N(j)|} \left[(1 - x_n^{j[\tau]}) + 2 \left(\frac{1}{\alpha_n^{j[\tau]}} - 1 \right) \eta_n^{j[\tau]} \right] D_n^{j[\tau]} (* \underline{\underline{\omega}}_n^{j[\tau]})$$

$$+ \sum_{i_2=1}^{|I_2|} g_{i_2}^{j[\tau]} \alpha_{i_2}^{[\tau]} [(1 - x_{i_2}'^{[\tau]}) + 2(1 - \alpha_{i_2}^{[\tau]}) \eta_{i_2}^{[\tau]}] D_{i_2}^{[\tau]} (* \bar{\omega}_{i_2}^{[\tau]})$$

$$+ \sum_{i_2=1}^{|I_2|} (1 - `x_{i_2}^{[\tau]}) f_{i_2}^{j[\tau]} D_{i_2}^{[\tau]} (* \bar{\omega}_{i_2}^{[\tau]})$$

$$+ \sum_{i_2=1}^{|I_2|} \hat{g}_{i_2}^{j[\tau]} (1 - \alpha_{i_2}^{[\tau]}) [(1 - x_{i_2}'^{[\tau]}) + 2 \alpha_{i_2}^{[\tau]} \eta_{i_2}^{[\tau]}] D_{i_2}^{[\tau]} (* \bar{\omega}_{i_2}^{[\tau]}) \quad (34)$$

Inspection of Eq. 34 reveals that the contribution of each process to the storage size appears in a separate term. For example, the storage volume attributed to a type 1 production process is

$$\Bigg[(1 - `x_{i_1}^{[\tau]}) f_{i_1}^{j[\tau]} + (1 - x_{i_1}'^{[\tau]}) g_{i_1}^{j[\tau]} + 2(f_{i_1}^{j[\tau]} + g_{i_1}^{j[\tau]}) \left(\frac{1}{\alpha_{i_1}^{[\tau]}} - 1 \right) \eta_{i_1}^{[\tau]} \Bigg] D_{i_1}^{[\tau]} (* \underline{\underline{\omega}}_{i_1}^{[\tau]}) \quad (35)$$

The optimal objective value is

$$ATC^{[\tau]}(D_k^{j[\tau]}, D_i^{[\tau]}, D_n^{j[\tau]}, v^{j[\tau]}(0))$$
$$= \sum_{j=1}^{|J|} \sum_{k=1}^{|K(j)|} P_k^{j[\tau]} D_k^{j[\tau]} + 2 \sum_{j=1}^{|J|} \sum_{k=1}^{|K(j)|} \alpha_k^{j[\tau]} \sqrt{A_k^{j[\tau]} \Psi_k^{j[\tau]} D_k^{j[\tau]}}$$
$$+ \sum_{j=1}^{|J|} \sum_{n=1}^{|N(j)|} P_n^{j[\tau]} D_n^{j[\tau]} + 2 \sum_{j=1}^{|J|} \sum_{n=1}^{|N(j)|} \alpha_n^{j[\tau]} \sqrt{A_n^{j[\tau]} \Psi_n^{j[\tau]} D_n^{j[\tau]}}$$
$$+ 2 \sum_{i_1=1}^{|I_1|} \alpha_{i_1}^{[\tau]} \sqrt{A_{i_1}^{[\tau]} \Psi_{i_1}^{[\tau]} D_{i_1}^{[\tau]}} + 2 \sum_{i_2=1}^{|I_2|} \sqrt{A_{i_2}^{[\tau]} \Psi_{i_2}^{[\tau]} D_{i_2}^{[\tau]}}$$
$$+ \sum_{j=1}^{|J|} \sum_{m=1}^{|M(j)|} [(0.5 H^{j[\tau]} + b^{j[\tau]})(1 - x_m^{j[\tau]}) \underline{\underline{\omega}}_m^{j[\tau]} + (H^{j[\tau]}$$
$$+ 2 b^{j[\tau]}) d_m^{j[\tau]} - P_m^{j[\tau]}] D_m^{j[\tau]}$$
$$+ 0.5 \sum_{j=1}^{|j|} H^{j[\tau]} (v^{j[\tau]}(0) + v^{j[\tau+1]}(0)) + \sum_{j=1}^{|J|} b^{j[\tau]} \max\{v^{j[\tau]}$$
$$\times (0), v^{j[\tau+1]}(0)\} \quad (36)$$

The optimal objective, Eq. 36, has separate cost terms for each process. For example, the cost of a type 2 process is

$$\text{Cost of process } i_2 = 2 \sqrt{A_{i_2}^{[\tau]} \Psi_{i_2}^{[\tau]} D_{i_2}^{[\tau]}} \quad (37)$$

In the derivation of the above optimal solutions, the average flow rates $D_k^{j[\tau]}$, $D_i^{[\tau]}$, $D_n^{j[\tau]}$, and $D_m^{j\,[\tau]}$ are assumed to be constants; however, they are usually variables. The optimal average flow rates can be calculated by solving another optimization problem whose objective function is Eq. 36 and constraints are Eqs. 18 and 19; this is called the second-level optimization problem and is summarized in Table 2. The global optimality for such a decomposition of the original optimization problem is proved in Appendix A of our previous study.[13] The nonlinear objective function in Eq. 36 is a separable concave function and can be linearized in a piecewise process by using the specially ordered sets (SOS) formulation.[14] The noncontinuous term can be replaced with a linear variable by adding the following inequality constraints:

$$\max\{v^{j[\tau]}(0), v^{j[\tau+1]}(0)\} \geq v^{j[\tau]}(0)$$
$$\max\{v^{j[\tau]}(0), v^{j[\tau+1]}(0)\} \geq v^{j[\tau+1]}(0) \quad (38)$$

Table 2. Second-Level Problem

Minimize $TC = \sum_{\tau=0}^{T-1} ATC^{[\tau]} \nabla t^{[\tau]} \prod_{\tau'=0}^{\tau-1} (1 + \rho^{[\tau']})^{-1} - \sum_{j=1}^{|J|} h^{j[T]} \upsilon^{j[T]}(0)$

with respect to $(D_k^{j[\tau]}, D_i^{[\tau]}, D_n^{j[\tau]}, D_m^{j[\tau]}$ and $\upsilon^{j[\tau]}(0)$

such that

$$\dot{\upsilon}^{j[\tau]} = \sum_{i_1=1}^{|I_1|} g_{i_1}^{j[\tau]} D_{i_1}^{[\tau]} + \sum_{i_2=1}^{|I_2|} (g_{i_2}^{j[\tau]} \alpha_{i_2}^{[\tau]} + \hat{g}_{i_2}^{j[\tau]}(1 - \alpha_{i_2}^{[\tau]})) D_{i_2}^{[\tau]} + \sum_{k=1}^{|K(j)|} D_k^{j[\tau]} - \sum_{i_1=1}^{|I_1|} f_{i_1}^{j[\tau]} D_{i_1}^{[\tau]} - \sum_{i_2=1}^{|I_2|} f_{i_2}^{j[\tau]} D_{i_2}^{[\tau]} - \sum_{m=1}^{|M(j)|} D_m^{j[\tau]} - \sum_{n=1}^{|N(j)|} D_n^{j[\tau]}$$

$$\upsilon^{j[\tau+1]}(0) = \upsilon^{j[\tau]}(0) + \dot{\upsilon}^{j[\tau]} \nabla t^{[\tau]}, \; \upsilon^{j[0]}(0) = \upsilon_0^j, \; \underline{\underline{\upsilon}}_T^j \le \upsilon^{j[T]}(0) \le \overline{\overline{\upsilon}}_T^j, \; \underline{\underline{D}}_m^{j[\tau]} \le D_m^{j[\tau]} \le \bar{\bar{D}}_m^{j[\tau]}$$

where

$$ATC^{[\tau]} = \sum_{j=1}^{|J|} \sum_{k=1}^{|K(j)|} P_k^{j[\tau]} D_k^{j[\tau]} + 2 \sum_{j=1}^{|J|} \sum_{k=1}^{|K(j)|} \alpha_k^{j[\tau]} \sqrt{A_k^{j[\tau]} \Psi_k^{j[\tau]} D_k^{j[\tau]}} + \sum_{j=1}^{|J|} \sum_{n=1}^{|N(j)|} P_n^{j[\tau]} D_n^{j[\tau]} + 2 \sum_{j=1}^{|J|} \sum_{n=1}^{|N(j)|} \alpha_n^{j[\tau]} \sqrt{A_n^{j[\tau]} \Psi_n^{j[\tau]} D_n^{j[\tau]}} + 2 \sum_{i_1=1}^{|I_1|} \alpha_{i_1}^{[\tau]} \sqrt{A_{i_1}^{[\tau]} \Psi_{i_1}^{[\tau]} D_{i_1}^{[\tau]}} + 2 \sum_{i_2=1}^{|I_2|} \sqrt{A_{i_2}^{[\tau]} \Psi_{i_2}^{[\tau]} D_{i_2}^{[\tau]}}$$

$$+ \sum_{j=1}^{|J|} \sum_{m=1}^{|M(j)|} [(0.5H^{j[\tau]} + b^{j[\tau]})(1 - x_m^{j[\tau]}) \underline{\underline{\omega}}_m^{j[\tau]} + (H^{j[\tau]} + 2b^{j[\tau]}) d_m^{j[\tau]} - P_m^{j[\tau]}] D_m^{j[\tau]} + 0.5 \sum_{j=1}^{|j|} H^{j[\tau]} (\upsilon^{j[\tau]}(0) + \upsilon^{j[\tau+1]}(0)) + \sum_{j=1}^{|J|} b^{j[\tau]} \max\{\upsilon^{j[\tau]}(0), \upsilon^{j[\tau+1]}(0)\}$$

$\max\{\upsilon^{j[\tau]}(0), \; \upsilon^{j[\tau+1]}(0)\} \ge \upsilon^{j[\tau+1]}(0), \; \max\{\upsilon^{j[\tau]}(0), \; \upsilon^{j[\tau+1]}(0)\} \ge \upsilon^{j[\tau]}(0)$

The overall computation procedure is the same as in our previous work.[14] First, the optimal average flow rates should be obtained by solving the second-level problem in Table 2. Then, analytical solutions of the cycle times, batch sizes, storage sizes, and initial start-up times can be calculated by using Eqs. 25–34.

We summarize the major assumptions introduced in this study:

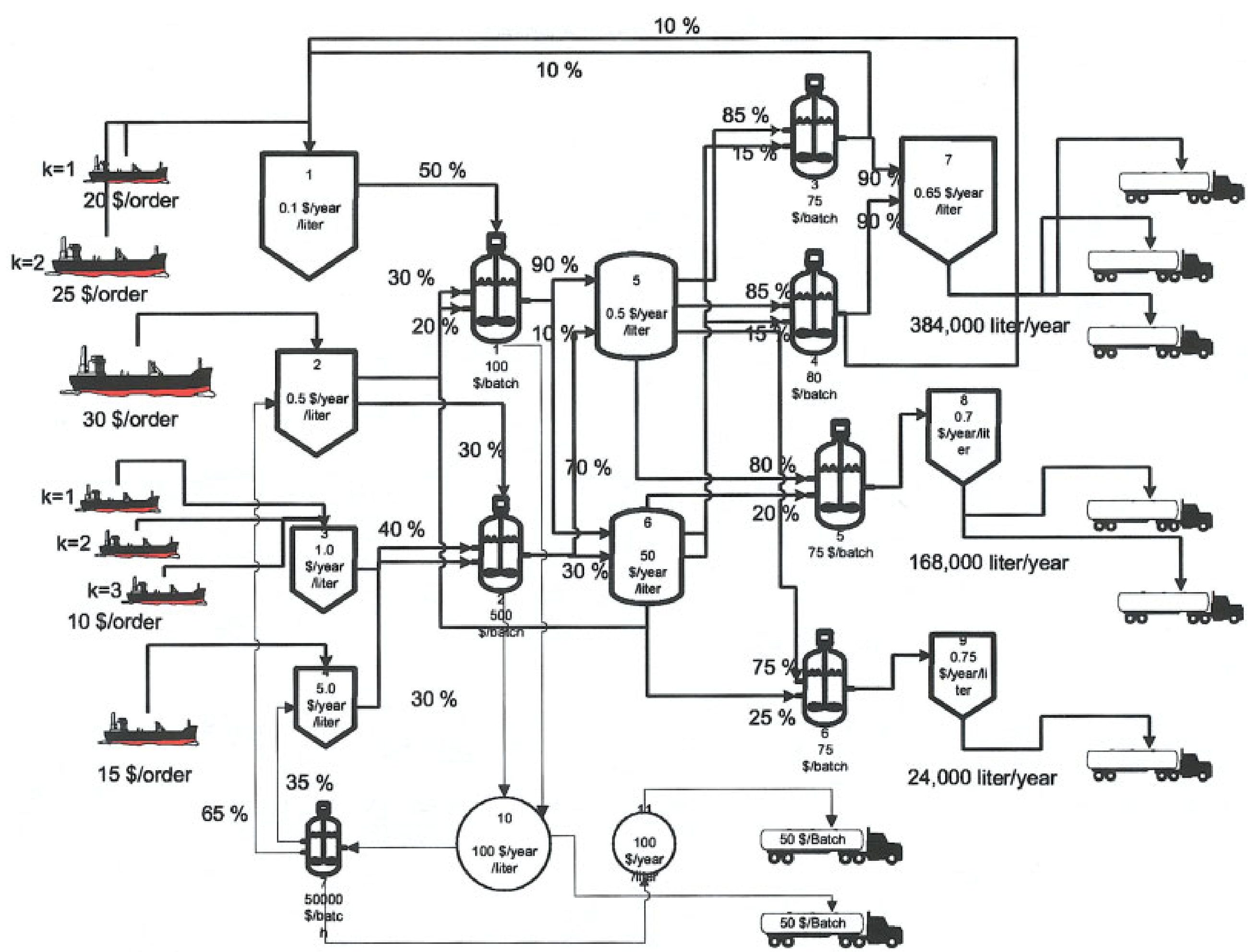

Figure 10. Example plant design problem.

[Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com]

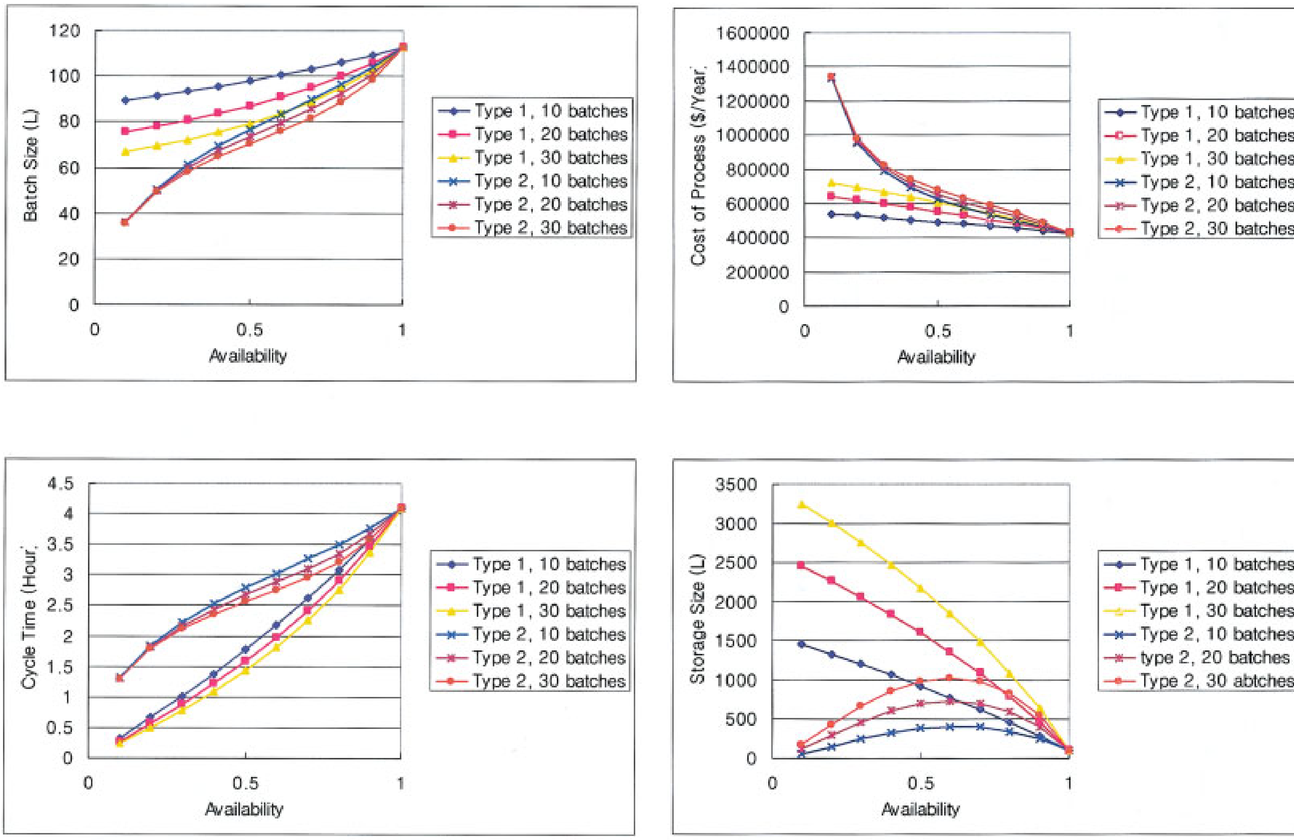

Figure 11. Variation of optimal solutions with respect to availability.

[Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com]

(1) The batch size of type 1 process is deterministic but its cycle time is uncertain.

(2) The cycle time of type 2 process is deterministic but its batch size is uncertain.

(3) $\alpha_{i_2}\eta_{i_2}$ is an integer.

(4) Average flow rates and all parameters are constant within a period.

(5) The length of a time period is sufficiently longer than batch cycle times.

(6) The average accumulated flow is equidistant from the upper and lower bounds of the flow.

(7) α and η are known parameters.

Example Plant Design

Let us consider a plant that produces three finished products from four raw materials. The configuration of this plant is shown in Figure 10, which also includes most of the input data required for the computations. A similar plant configuration, but without process I7 and storage units J10 and J11, was studied previously.[13] Processes I1 and I2 are type 2 processes; the waste materials of failed batches are collected in storage unit J10. The waste material J10 can be disposed of by the waste disposal process J10 or can be regenerated through process I7 to raw materials J2 and J4. The process I7 is also a type 2 process; waste material from this process goes to storage unit J11. The waste material J11 is disposed of by the waste disposal process J11. All of the other processes are type 1 processes. (Note that J10 and J11 are either of storage unit index, waste material index, or waste disposal process index.)

Figure 11 shows the dependencies of the optimal solutions for batch size, cycle time, cost of process, and storage size, on availability. Equations 29, 31, 34, and 36 and the data of process I1 were used in the computation. Six cases were considered, depending on the process type and the number of batches in the long cycle time. Calculations were performed for batch numbers of η_i = 10, 20, and 30. All optimal solutions return to the solutions of the deterministic version of the model when the availability is 1. As the availability approaches 1, the cost of each process decreases, whereas the batch size and cycle time increase. Processes of types 1 and 2 show different graph patterns. In particular, the cycle time and cost of the type 2 processes are greater than those of the type 1 processes, whereas the batch size and storage size of the type 2 processes are smaller than those of the type 1 processes. Increasing η_i reduces the batch size and cycle time but increases the cost of the process and the storage size. For all variables except storage size, changing η_i has only a slight effect compared to varying the availability. The storage size of each type 2 process exhibits a maximum with respect to availability. When the availability is greater than that of the maximum storage size, the storage size increases because the uncertainty effect is dominant. As the availability goes below that of the maximum storage size, the on-spec product flow to the storage significantly decreases and, consequently, the storage size decreases in spite of considerable uncertainty.

Multiperiod optimization of the second-level problem was conducted with 10 periods, constant demand, a 1.5-fold increase in the raw material price at period T4, and a 0.5-fold decrease in the raw material price at period T7. The resulting average flow rate and inventory change of raw material J2 are plotted in Figure 12. Inventory shows maxima at T3 and T7, which is typical for a planning model. The availabilities were set to 0.9 for all processes except process I7, which was assigned an availability of 0.4. The setup cost and capital cost of process I7 were set to more than 100-fold those of the other processes. The disposal price of waste J11 was set to $2000/L and the disposal price of waste J10 was changed from $300/L to $1200/L. As shown in Figure 13, the disposal of waste J10 is more economical than the regeneration of this waste through process I7 until period T7. After period T7, regeneration through process I7 is more economical than disposal because of the increased price of disposal. However, adding the regeneration process was very economical considering that the capital and setup costs of process I7 were 100-fold higher than those of the other processes in this simulation.

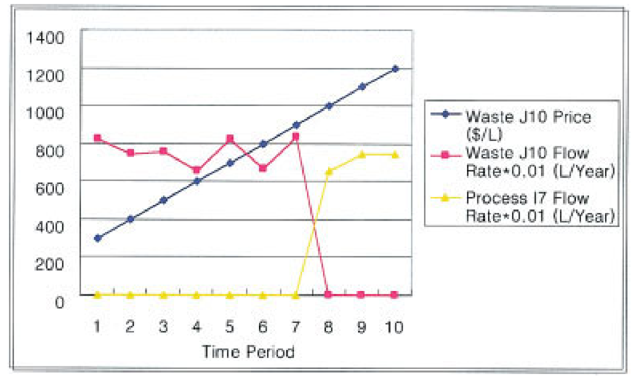

Figure 13. Selection of waste treatment method with respect to waste disposal price change.

[Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com]

Concluding Remarks

The methodology presented here determines the optimal sizes of batch processes and storage units interconnected in a network structure when the processes are subject to random failures of operating time and material spoilage. The main sources of random failures were operating time losses and batch material losses. Waste regeneration and disposal processes were installed to treat materials from failed batches. The PSW model was judiciously used to find the upper and lower bounds of flows susceptible to short-term random variations in the cycle time or batch size. A multiperiod formulation was combined with the PSW model to account for the long-term trend in product demand. In the definition of the random properties, availability and number of batches in a long cycle time were introduced as input parameters instead of more widely used parameters such as the mean and variance. The availability is commonly used in process reliability analysis methods such as FMEA and the number of batches in a long cycle time is proportional to the variance. These parameters were more practical and easier to estimate based on human perception. The optimization problem consisted of minimizing the sum of the setup cost, capital cost of processes, inventory holding cost, and price of materials under the constraints of meeting random product demand and no depletion of materials.

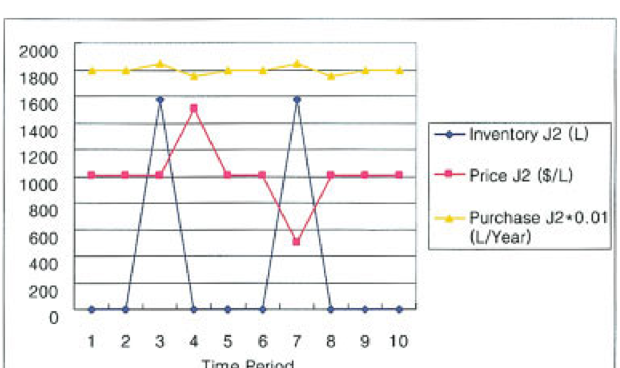

Figure 12. Flow rate and inventory change of raw material with respect to price change.

[Color figure can be viewed in the online issue, which is available at www.interscience.wiley.com]

The PSW model with judicious graphical analysis of accumulated flows provided great flexibility to accommodate random variations of various types of material flows into one optimization formulation, resulting in analytical solutions. These analytical solutions substantially reduce the computational burden, which is the major achievement of this study. The variables that could not be solved analytically were the average flow rates. The concave cost minimization network flow problem was solved to obtain the optimal average flow rates. The computation time to solve this problem with a well-known algorithm was about sevenfold that required to solve the linear programming problem of the same size. Even though the average flow rates can be obtained by other methods such as the ordinary planning model, the optimality of other variables is still valid. The analytical optimal solutions made it possible to conduct a sensitivity analysis with respect to the input parameters, availability, and number of batches in a long cycle time.

Acknowledgments

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Notation

a_k^j, $a_k^{j[\tau]}$ = annualized capital cost of raw material purchasing facility, dollars per unit of item per year

a_{i}, $a_i^{[\tau]}$ = annualized capital cost of unit i, dollars per unit of item per year

a_n^j, $a_n^{j[\tau]}$ = annualized capital cost of waste disposal facility, dollars per unit of item per year

b^{j}, $b^{\mathrm{j}[\tau]}$ = annualized capital cost of storage facility, dollars per unit of item per year

A_k^j, $A_k^{j[\tau]}$ = ordering cost of feedstock materials, dollar per order

A_{i}, $A_i^{[\tau]}$ = ordering cost of noncontinuous units, dollar per order

A_n^j, $A_n^{j[\tau]}$ = disposal cost of waste materials, dollar per order

ATC, $ATC^{[\tau]}$ = annualized sum of costs at time period τ, $/year

B = general description of batch size of any process, units per lot

$\bar{B}$ = general description of average batch size of any process, units per lot

B_k^j = raw material order size, units per lot

$\bar{B}_k^j$, $\bar{B}_k^{j[\tau]}$ = mean of raw material order size, units per lot
B_i = batch production unit size, units per lot
$\bar{B}_i$, $\bar{B}_i^{[\tau]}$ = mean of batch production unit size, units per lot
$\mathbf{B}'_{i_2(l)}$ = random lth batch size of product flow of type 2 process, units per lot
$\mathbf{B}_m^j$, $\mathbf{B}_{m(l)}^j$ = random lth batch size of demand flow of product j, customer m, units per lot
B_m^j = final product delivery batch size, units per lot
$\bar{\bar{B}}_m^j$, $\bar{\bar{B}}_m^{j[\tau]}$ = maximum of final product demand batch size, units per lot
B_n^j, $B_n^{j[\tau]}$ = batch size of waste disposal, units per lot
d = general description of failure duration within a long cycle time, year
d_k^j, $d_k^{j[\tau]}$ = total failure duration within a long cycle time for raw material purchasing process, year
d_i, $d_i^{[\tau]}$ = total failure duration within a long cycle time for process i, year
d_m^j, $d_m^{j[\tau]}$ = total failure duration within a long cycle time for finished product demand, year
d_n^j, $d_n^{j[\tau]}$ = total failure duration within a long cycle time for waste disposal process, year
$\hat{d}_{i_2}$ = total failure duration within a long cycle time for waste flow of type 2 process i, year
D = general description of average flow rates of any process, units per year
D_i, $D_i^{[\tau]}$ = average material flow rate through batch production units, units per year
D_k^j, $D_k^{j[\tau]}$ = average material flow rate of raw material supply, units per year
D_m^j, $D_m^{j[\tau]}$ = average material flow rate of customer demand, units per year
D_n^j, $D_n^{j[\tau]}$ = average material flow rate of waste disposal, units per year
f_i^j, $f_i^{j[\tau]}$ = feedstock composition of unit i at time period τ
$\mathbf{F}_1^{(t)}$ = accumulated flow of type 1 random process, units
$F_{N_2}^{j(t)}$ = accumulated feed flow of type 2 process, units
g = general description of product yield
g_i^j, $g_i^{j[\tau]}$ = product yield of process i at time period τ
$\hat{g}$ = general description of waste material yield
$\hat{g}_{i2}$, $\hat{g}_{i2}^{j[\tau]}$ = waste material yield of type 2 process i_2 at time period τ
$\mathbf{G}_{i_2}^j(t)$ = random product flow of type 2 process i_2
$\hat{\mathbf{G}}_{i_2}^j(t)$ = random waste flow of type 2 process i_2
$h^{j[T]}$ = salvage cost of ending inventory, dollars per unit of item
H^j, $H^{j[\tau]}$ = annual inventory holding costs, dollars per unit of item per year
I = noncontinuous process set
I_1 = noncontinuous process subset of type 1
I_2 = noncontinuous process subset of type 2
J = storage or material set
$K(j)$ = raw material supplier set for material j
$M(j)$ = consumer set for material j
$N(j)$ = waste disposal sink set for material j
P_k^j, $P_k^{j[\tau]}$ = raw material purchase price, dollars per unit of item
P_m^j, $P_m^{j[\tau]}$ = finished product sales price, dollars per unit of item
P_n^j, $P_n^{j[\tau]}$ = waste disposal price, dollars per unit of item
t_m^j, $t_m^{j[\tau]}$ = start-up time of customer demand
t_n^j, $t_n^{j[\tau]}$ = start-up time of waste disposal
$`t_i$, $`t_i^{[\tau]}$ = start-up time of feedstock feeding to batch process i
t'_i, $t_i'^{[\tau]}$ = start-up time of product discharging from batch process i
t_k^j, $t_k^{j[\tau]}$ = start-up time of raw material purchasing
Δt_i, $\Delta t_i^{[\tau]}$ = time delay between feed flow and product flow of process i, year
$\nabla t^{[\tau]}$ = time period interval, year
T = ending time period
TC = total cost, \$/year
$\bar{\bar{V}}^j$, $\bar{\bar{V}}^{j[\tau]}$ = upper bound of inventory holdup, units of item
$\underline{V}^j$, $\underline{V}^{j[\tau]}$ = lower bound of inventory holdup, units of item
$V^j(t)$, $V^{j[\tau]}(t)$ = inventory holdup, units of item
$V^j(0)$, $V^{j[\tau]}(0)$ = initial inventory holdup, units of item
$\bar{V}^j$, $\bar{V}^{j[\tau]}$ = time-averaged inventory holdup, units of item
$\dot{v}^{j[\tau]}$ = rate of multiperiod inventory, units per year
$\bar{\bar{v}}^{j[\tau]}$ = upper bound of multiperiod inventory, units of item
$\bar{v}^{j[\tau]}$ = average of multiperiod inventory, units of item
$\underline{v}^{j[\tau]}$ = lower bound of multiperiod inventory, units of item
$v^{j[\tau]}(0)$ = initial value of multiperiod inventory, units of item
$\underline{\underline{v}}_T^j$ = lower bound of ending multiperiod inventory, units of item
$\bar{\bar{v}}_T^j$ = upper bound of ending multiperiod inventory, units of item
x_k^j, $x_k^{j[\tau]}$ = storage operation time fraction of purchasing raw materials
$`x_i$, $`x_i^{[\tau]}$ = storage operation time fraction of feeding to noncontinuous unit i
x' = general description of storage operation time fraction of discharging flow
x'_i, $x_i'^{[\tau]}$ = storage operation time fraction of discharging from noncontinuous unit i
x_m^j, $x_m^{j[\tau]}$ = storage operation time fraction of finished product demand
x_n^j, $x_n^{j[\tau]}$ = storage operation time fraction of waste disposal flow

Greek letters

α = general description of availability
α_k^j, $\alpha_k^{j[\tau]}$ = availability of raw material purchase flow
α_i, $\alpha_i^{[\tau]}$ = availability of process i
α_n^j, $\alpha_n^{j[\tau]}$ = availability of waste disposal flow
δ_1, δ_2, δ_3, δ_4 = confidence limits
ε_1 = convergence limit
γ_m^j = minimum number of batch in long cycle time of customer demand
$\lambda^{j[\tau]}$ = Lagrangian multiplier of Kuhn–Tucker conditions
η = general description of number of batch in long cycle time
η_k^j, $\eta_k^{j[\tau]}$ = number of batch in long cycle time for purchasing material flow
η_i, $\eta_i^{[\tau]}$ = number of batch in long cycle time for processing material flow
η_m^j, $\eta_m^{j[\tau]}$ = number of batch in long cycle time for finished product demand flow
η_n^j, $\eta_n^{j[\tau]}$ = number of batch in long cycle time for waste disposal flow
$\rho^{[\tau]}$ = discount rate of τ period to compute net present value, \$/\$
ω_m^j, $\omega_m^{j[\tau]}$ = cycle time of customer demand, year
ω_n^j, $\omega_n^{j[\tau]}$ = cycle time of waste disposal flow, year
ω_k^j, $\omega_k^{j[\tau]}$ = cycle time of raw material purchasing, year
ω_i, $\omega_i^{[\tau]}$ = cycle time of process i, year
$\boldsymbol{\omega}_{(l)}$ = lth random cycle time of any process, year
$\boldsymbol{\omega}_{i_2}$ = random cycle time of type 2 process i_2, year
$\boldsymbol{\omega}_m^j$, $\boldsymbol{\omega}_{m(l)}^j$ = (lth) random cycle time of customer demand, year
$\bar{\omega}$ = general description of average cycle time, year
$\bar{\omega}_{i_2}$, $\bar{\omega}_{i_2}^{[\tau]}$ = average cycle time of type 2 process i_2, year
$\underline{\underline{\omega}}$ = general description of lower bound of cycle time, year
$\underline{\underline{\omega}}_k^j$, $\underline{\underline{\omega}}_k^{j[\tau]}$ = lower bound of cycle time of raw material purchase, year
$\underline{\underline{\omega}}_m^j$, $\underline{\underline{\omega}}_m^{j[\tau]}$ = lower bound of cycle time of finished product demand flow, year
$\underline{\underline{\omega}}_n^j$, $\underline{\underline{\omega}}_n^{j[\tau]}$ = lower bound of cycle time of waste disposal flow, year
$\tilde{\omega}$ = general description of long cycle time, year
$\tilde{\omega}_i$, $\tilde{\omega}_i^{[\tau]}$ = long cycle time of process i, year
$\tilde{\omega}_k^j$, $\tilde{\omega}_k^{j[\tau]}$ = long cycle time of raw material purchasing, year
$\tilde{\omega}_m^j$, $\tilde{\omega}_m^{j[\tau]}$ = long cycle time of customer demand, year
$\tilde{\omega}_n^j$, $\tilde{\omega}_n^{j[\tau]}$ = long cycle time of waste disposal flow, year
$\Psi_k^{j[\tau]}$ = aggregated cost defined by Eq. 26
$\Psi_n^{j[\tau]}$ = aggregated cost defined by Eq. 28
$\Psi_{i_1}^{[\tau]}$ = aggregated cost defined by Eq. 30
$\Psi_{i_2}^{[\tau]}$ = aggregated cost defined by Eq. 32

Subscripts

- i = batch production process index
- i_1 = type 1 process index
- i_2 = type 2 process index
- k = index of raw material vendors
- (l) = batch sequence index
- m = index of finished product customers
- n = index of waste disposal sinks

Superscripts

- j = storage index
- $[\tau]$ = time period index

Special functions

- $\mathrm{int}[\cdot]$ = truncation function to make integer
- $\mathrm{res}[\cdot]$ = positive residual function to be truncated
- $\mathrm{Var}(\cdot)$ = variance
- $|X|$ = number of elements in set X
- $P\{\cdot\}$ = probability
- $\underline{X}$ = average of X
- $\bar{\bar{X}}$ = upper bound of X
- $\underline{\underline{X}}$ = lower bound of X

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Appendix A: Kuhn–Tucker Solution to the Multiperiod Problem

The single period objective function Eq. 16 can be rewritten as

$$
\begin{aligned}
ATC^{[\tau]} &= \sum_{j=1}^{|J|}\sum_{k=1}^{|K(j)|}\frac{\alpha_k^{j[\tau]}A_k^{j[\tau]}}{\underline{\underline{\omega}}_k^{j[\tau]}} + \sum_{j=1}^{|J|}\sum_{k=1}^{|K(j)|}P_k^{j[\tau]}D_k^{j[\tau]} \\
&\quad + \sum_{j=1}^{|J|}\sum_{n=1}^{|N(j)|}\left[\frac{\alpha_n^{j[\tau]}A_n^{j[\tau]}}{\underline{\underline{\omega}}_n^{j[\tau]}} + P_n^{j[\tau]}D_n^{j[\tau]}\right] \\
&\quad + \sum_{i_1=1}^{|I_1|}\frac{\alpha_{i_1}^{[\tau]}A_{i_1}^{[\tau]}}{\underline{\underline{\omega}}_{i_1}^{[\tau]}} + \sum_{i_2=1}^{|I_2|}\left[\frac{A_{i_2}^{[\tau]}}{\bar{\omega}_{i_2}^{[\tau]}} + a_{i_2}^{[\tau]}\left(\frac{D_{i_2}^{[\tau]}\bar{\omega}_{i_2}^{[\tau]}}{\alpha_{i_2}^{[\tau]}}\right)\right] \\
&\quad - \sum_{j=1}^{|J|}\sum_{n=1}^{|N(j)|}\left[0.5H^{j[\tau]}(1 - x_n^{j[\tau]}) - \frac{a_n^{j[\tau]}}{\alpha_n^{j[\tau]}}\right]D_n^{j[\tau]}\underline{\underline{\omega}}_n^{j[\tau]} + \sum_{j=1}^{|J|}\sum_{n=1}^{|M(j)|}(H^{j[\tau]} \\
&\quad + b^{j[\tau]})D_n^{j[\tau]}t_n^{j[\tau]} \\
&\quad + \sum_{j=1}^{|J|}\sum_{k=1}^{|K(j)|}\left(0.5H^{j[\tau]}(1 - x_k^{j[\tau]}) + b^{j[\tau]}(1 - x_k^{j[\tau]}) + \frac{a_k^{j[\tau]}}{\alpha_k^{j[\tau]}}\right)D_k^{j[\tau]}\underline{\underline{\omega}}_k^{j[\tau]} \\
&\quad - \sum_{j=1}^{|J|}\sum_{k=1}^{|K(j)|}(H^{j[\tau]} + b^{j[\tau]})D_k^{j[\tau]}t_k^{j[\tau]} + \sum_{j=1}^{|J|}\sum_{i_1=1}^{|I_1|}(H^{j[\tau]} + b^{j[\tau]}) \\
&\quad \times (f_{i_1}^{j[\tau]} - g_{i_1}^{j[\tau]})D_{i_1}^{[\tau]}{}^{\backprime}t_{i_1}^{[\tau]} \\
&\quad + \left(\sum_{i_1=1}^{|I_1|}\frac{a_{i_1}^{[\tau]}}{\alpha_{i_1}^{[\tau]}} + \sum_{j=1}^{|J|}\sum_{i_1=1}^{|I_1|}0.5H^{j[\tau]}[(1 - x_{i_1}'^{[\tau]})g_{i_1}^{j[\tau]}\right. \\
&\quad \left. - (1 - {}^{\backprime}x_{i_1}^{[\tau]})f_{i_1}^{j[\tau]}]\right)D_{i_1}^{[\tau]}\underline{\underline{\omega}}_{i_1}^{[\tau]} \\
&\quad - \sum_{j=1}^{|J|}(H^{j[\tau]} + b^{j[\tau]})\sum_{i_1=1}^{|I_1|}g_{i_1}^{j[\tau]}D_{i_1}^{[\tau]}\Delta t_{i_1}^{[\tau]} \\
&\quad + \sum_{j=1}^{|J|}b^{j[\tau]}\sum_{i_1=1}^{|I|}g_{i_1}^{j[\tau]}D_{i_1}^{[\tau]}(1 - x_{i_1}'^{[\tau]})\underline{\underline{\omega}}_{i_1}^{[\tau]} \\
&\quad + \sum_{j=1}^{|J|}\sum_{i_2=1}^{|I_2|}(H^{j[\tau]} + b^{j[\tau]})(f_{i_2}^{j[\tau]} - g_{i_2}^{j[\tau]}\alpha_{i_2}^{[\tau]} - \hat{g}_{i_2}^{j[\tau]}(1 - \alpha_{i_2}^{[\tau]}))D_{i_2}^{[\tau]}{}^{\backprime}t_{i_2}^{[\tau]} \\
&\quad - \sum_{j=1}^{|J|}(H^{j[\tau]} + b^{j[\tau]})\sum_{i_2=1}^{|I_2|}(g_{i_2}^{j[\tau]}\alpha_{i_2}^{[\tau]} + \hat{g}_{i_2}^{j[\tau]}(1 - \alpha_{i_2}^{[\tau]}))D_{i_2}^{[\tau]}\Delta t_{i_2}^{[\tau]} \\
&\quad + \sum_{j=1}^{|J|}\sum_{i_2=1}^{|I_2|}0.5H^{j[\tau]}g_{i_2}^{j[\tau]}\alpha_{i_2}^{[\tau]}(1 - x_{i_2}'^{[\tau]})D_{i_2}^{[\tau]}\bar{\omega}_{i_2}^{[\tau]} \\
&\quad + \sum_{j=1}^{|J|}\sum_{i_2=1}^{|I_2|}0.5H^{j[\tau]}\hat{g}_{i_2}^{j[\tau]}(1 - \alpha_{i_2}^{[\tau]})(1 - x_{i_2}'^{[\tau]})D_{i_2}^{[\tau]}\bar{\omega}_{i_2}^{[\tau]} \\
&\quad + \sum_{i_2=1}^{|I_2|}b^{j[\tau]}g_{i_2}^{j[\tau]}\alpha_{i_2}^{[\tau]}((1 - x_{i_2}'^{[\tau]}) + (1 - \alpha_{i_2}^{[\tau]})\eta_{i_2}^{[\tau]})D_{i_2}^{[\tau]}\bar{\omega}_{i_2}^{[\tau]} \\
&\quad + \sum_{i_2=1}^{|I_2|}b^{j[\tau]}\hat{g}_{i_2}^{j[\tau]}(1 - \alpha_{i_2}^{[\tau]})((1 - x_{i_2}'^{[\tau]}) + \alpha_{i_2}^{[\tau]}\eta_{i_2}^{[\tau]})D_{i_2}^{[\tau]}\bar{\omega}_{i_2}^{[\tau]}
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{j=1}^{|J|}\sum_{i_2=1}^{|I_2|} 0.5H^{j[\tau]}(1-{}^{\backprime}x_{i_2}^{[\tau]})\, f_{i_2}^{[\tau]} D_{i_2}^{[\tau]} \bar{\omega}_{i_2}^{[\tau]} \\
&+ \sum_{j=1}^{|J|}\sum_{n=1}^{|N(j)|} b^{j[\tau]} D_n^{j[\tau]}\left(\frac{1}{\alpha_n^{j[\tau]}}-1\right)\eta_n^{j[\tau]}\underline{\underline{\omega}}_n^{j[\tau]} \\
&+ \sum_{j=1}^{|J|}\sum_{k=1}^{|K(j)|} b^{j[\tau]} D_k^{j[\tau]}\left(\frac{1}{\alpha_k^{j[\tau]}}-1\right)\eta_k^{j[\tau]}\underline{\underline{\omega}}_k^{j[\tau]} \\
&+ \sum_{i_1=1}^{|I_1|}\sum_{j=1}^{|J|} b^{j[\tau]}(f_{i_1}^{j[\tau]}+g_{i_1}^{j[\tau]})\left(\frac{1}{\alpha_{i_1}^{[\tau]}}-1\right)\eta_{i_1}^{[\tau]}D_{i_1}^{[\tau]}\underline{\underline{\omega}}_{i_1}^{[\tau]} + \text{constant}
\end{aligned} \tag{A1}
$$

$$
\begin{aligned}
\text{constants} = {} & 0.5 \sum_{j=1}^{|J|} H^{j[\tau]}(v^{j[\tau]}(0)+v^{j[\tau+1]}(0)) + \sum_{j=1}^{|J|} b^{j[\tau]}\max\{v^{j[\tau]} \\
& \times (0), v^{j[\tau+1]}(0)\} + \sum_{j=1}^{|J|}(H^{j[\tau]}+b^{j[\tau]})V^{j[\tau]}(0) \\
& - \sum_{j=1}^{|J|}\sum_{m=1}^{|M(j)|} 0.5H^{j[\tau]}(1-x_m^{j[\tau]})\,D_m^{j[\tau]}\underline{\underline{\omega}}_m^{j[\tau]} + \sum_{j=1}^{|J|}(H^{j[\tau]} \\
& + b^{j[\tau]})\sum_{m=1}^{|M(j)|} D_m^{j[\tau]} t_m^{j[\tau]} \\
& - \sum_{j=1}^{|J|}\sum_{M=1}^{|M(j)|} P_m^{j[\tau]}D_m^{j[\tau]} + \sum_{j=1}^{|J|}\sum_{m=1}^{|M(j)|} b^{j[\tau]}D_m^{j[\tau]}d_m^{j[\tau]}
\end{aligned} \tag{A2}
$$

The Lagrangian for the optimization problem to minimize Eq. 23 subject to Eq. 24 with respect to $\underline{\underline{\omega}}_k^{j[\tau]}$, $\underline{\underline{\omega}}_{i_1}^{[\tau]}$, $\bar{\omega}_{i_2}^{[\tau]}$, $\underline{\underline{\omega}}_n^{j\,[\tau]}$ and $t_k^{j[\tau]}$, ${}^{\backprime}t_{i_1}^{[\tau]}$, ${}^{\backprime}t_{i_2}^{[\tau]}$, $t_n^{j\,[\tau]}$ is

$$
\begin{aligned}
L = {} & TC \\
& - \sum_{\tau=0}^{T-1}\sum_{j=1}^{|J|}\lambda^{j[\tau]}\Bigg[V^{j[\tau]}(0) - \sum_{k=1}^{|K(j)|} D_k^{j[\tau]}\left[t_k^{j[\tau]} + \left(\frac{1}{\alpha_k^{j[\tau]}}-1\right)\eta_k^{j[\tau]}\underline{\underline{\omega}}_k^{j[\tau]}\right] \\
& - \sum_{i_1=1}^{|I_1|} g_{i_1}^{j[\tau]} D_{i_1}^{[\tau]}\left[{}^{\backprime}t_{i_1}^{[\tau]} + \Delta t_{i_1}^{[\tau]} + \left(\frac{1}{\alpha_{i_1}^{[\tau]}}-1\right)\eta_{i_1}^{[\tau]}\underline{\underline{\omega}}_{i_1}^{[\tau]}\right] \\
& + \sum_{i_1=1}^{|I_1|} f_{i_1}^{j[\tau]} D_{i_1}^{[\tau]}\left[{}^{\backprime}t_{i_1}^{[\tau]} - (1-{}^{\backprime}x_{i_1}^{[\tau]})\underline{\underline{\omega}}_{i_1}^{[\tau]} - \left(\frac{1}{\alpha_{i_1}^{[\tau]}}-1\right)\eta_{i_1}^{[\tau]}\underline{\underline{\omega}}_{i_1}^{[\tau]}\right] \\
& - \sum_{i_2=1}^{|I_2|} g_{i_2}^{j[\tau]}\alpha_{i_2}^{[\tau]} D_{i_2}^{[\tau]}[{}^{\backprime}t_{i_2}^{[\tau]} + \Delta t_{i_2}^{[\tau]} + (1-\alpha_{i_2}^{[\tau]})\eta_{i_2}^{[\tau]}\bar{\omega}_{i_2}^{[\tau]}] \\
& - \sum_{i_2=1}^{|I_2|} \hat{g}_{i_2}^{j[\tau]}(1-\alpha_{i_2}^{[\tau]}) D_{i_2}^{[\tau]}[{}^{\backprime}t_{i_2}^{[\tau]} + \Delta t_{i_2}^{[\tau]} + \alpha_{i_2}^{[\tau]}\eta_{i_2}^{[\tau]}\bar{\omega}_{i_2}^{[\tau]}] \\
& + \sum_{i_2=1}^{|I_2|} f_{i_2}^{j[\tau]} D_{i_2}^{[\tau]}[{}^{\backprime}t_{i_2}^{[\tau]} - (1-{}^{\backprime}x_{i_2}^{[\tau]})\bar{\omega}_{i_2}^{[\tau]}] \\
& + \sum_{m=1}^{|M(j)|} D_m^{j[\tau]}[t_m^{j[\tau]} - (1-x_m^{j[\tau]})\underline{\underline{\omega}}_m^{j[\tau]} - d_m^{j[\tau]}] \\
& + \sum_{n=1}^{|N(j)|} D_n^{j[\tau]}\left[t_n^{j[\tau]} - (1-x_n^{j[\tau]})\underline{\underline{\omega}}_n^{j[\tau]} - \left(\frac{1}{\alpha_n^{j[\tau]}}-1\right)\eta_n^{j[\tau]}\underline{\underline{\omega}}_n^{j[\tau]}\right]\Bigg]
\end{aligned} \tag{A3}
$$

where $\lambda^{j[\tau]}$ is the Lagrangian multiplier. Kuhn–Tucker conditions give

$$\frac{\partial L}{\partial t_k^{j[\tau]}} = -(H^{j[\tau]}+b^{j[\tau]})\,D_k^{j[\tau]} + \lambda^{j[\tau]}D_k^{j[\tau]} = 0 \tag{A4}$$

$$
\begin{aligned}
\frac{\partial L}{\partial \underline{\underline{\omega}}_k^{j[\tau]}} = {} & -\frac{\alpha_k^{j[\tau]}A_k^{j[\tau]}}{(\underline{\underline{\omega}}_k^{j[\tau]})^2} + \Bigg[0.5H^{j[\tau]}(1-x_k^{j[\tau]}) + b^{j[\tau]}(1-x_k^{j[\tau]}) \\
& + \frac{a_k^{j[\tau]}}{\alpha_k^{j[\tau]}} + (b^{j[\tau]}+\lambda^{j[\tau]})\left(\frac{1}{\alpha_k^{j[\tau]}}-1\right)\eta_k^{j[\tau]}\Bigg]D_k^{j[\tau]} = 0
\end{aligned} \tag{A5}
$$

$$\frac{\partial L}{\partial t_n^{j[\tau]}} = (H^{j[\tau]}+b^{j[\tau]})\,D_n^{j[\tau]} - \lambda^{j[\tau]}D_n^{j[\tau]} = 0 \tag{A6}$$

$$
\begin{aligned}
\frac{\partial L}{\partial \underline{\underline{\omega}}_n^{j[\tau]}} = {} & -\frac{\alpha_n^{j[\tau]}A_n^{j[\tau]}}{(\underline{\underline{\omega}}_n^{j[\tau]})^2} + \Bigg[-0.5H^{j}(1-x_n^{j[\tau]}) + \frac{a_n^{j[\tau]}}{\alpha_n^{j[\tau]}} \\
& + \lambda^{j[\tau]}(1-x_n^{j[\tau]}) + (b^{j[\tau]}+\lambda^{j[\tau]})\left(\frac{1}{\alpha_n^{j[\tau]}}-1\right)\eta_n^{j[\tau]}\Bigg]D_n^{j[\tau]} = 0
\end{aligned} \tag{A7}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial t_{i_1}^{[\tau]}} = {} & \sum_{j=1}^{|J|}(H^{j[\tau]}+b^{j[\tau]})(f_{i_1}^{j[\tau]}-g_{i_1}^{j[\tau]})\,D_{i_1}^{[\tau]} \\
& - \sum_{j=1}^{|J|}\lambda^{j[\tau]}(f_{i_1}^{j[\tau]}-g_{i_1}^{j[\tau]})\,D_{i_1}^{[\tau]} = 0
\end{aligned} \tag{A8}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial \underline{\underline{\omega}}_{i_1}^{[\tau]}} = {} & -\frac{\alpha_{i_1}^{[\tau]}A_{i_1}^{[\tau]}}{(\underline{\underline{\omega}}_{i_1}^{[\tau]})^2} + \Bigg[\frac{a_{i_1}^{[\tau]}}{\alpha_{i_1}^{[\tau]}} + \sum_{j=1}^{|J|} 0.5H^{j[\tau]}[(1-x_{i_1}'^{[\tau]})\,g_{i_1}^{j[\tau]} \\
& - (1-{}^{\backprime}x_{i_1}^{[\tau]})\,f_{i_1}^{j[\tau]}]\Bigg]D_{i_1}^{[\tau]} \\
& + \sum_{j=1}^{|J|} b^{j[\tau]}g_{i_1}^{j[\tau]}(1-x_{i_1}'^{[\tau]})D_{i_1}^{[\tau]} - \sum_{j=1}^{|J|}(H^{j[\tau]}+b^{j[\tau]})\,g_{i_1}^{j[\tau]}D_{i_1}^{[\tau]}\frac{\partial \Delta t_{i_1}^{[\tau]}}{\partial \underline{\underline{\omega}}_{i_1}^{[\tau]}} \\
& + \sum_{J=1}^{|j|}(b^{j[\tau]}+\lambda^{j[\tau]})(f_{i_1}^{j[\tau]}+g_{i_1}^{j[\tau]})\left(\frac{1}{\alpha_{i_1}^{[\tau]}}-1\right)\eta_{i_1}^{[\tau]}D_{i_1}^{[\tau]} \\
& + \sum_{j=1}^{|J|}\lambda^{j[\tau]}\left[(1-{}^{\backprime}x_{i_1}^{[\tau]})\,f_{i_1}^{j[\tau]} + g_{i_1}^{j[\tau]}\frac{\partial \Delta t_{i_1}^{[\tau]}}{\partial \underline{\underline{\omega}}_{i_1}^{[\tau]}}\right]D_{i_1}^{[\tau]} = 0
\end{aligned} \tag{A9}
$$

$$\frac{\partial L}{\partial t_{i_2}^{[\tau]}} = \sum_{j=1}^{|J|} (H^{j[\tau]} + b^{j[\tau]})(f_{i_2}^{[\tau]} - g_{i_2}^{j[\tau]}\alpha_{i_2}^{[\tau]} - \hat{g}_{i_2}^{j[\tau]}(1 - \alpha_{i_2}^{[\tau]}))\, D_{i_2}^{[\tau]}$$

$$- \sum_{j=1}^{|J|} \lambda^{j[\tau]}(f_{i_2}^{[\tau]} - g_{i_2}^{j[\tau]}\alpha_{i_2}^{[\tau]} - \hat{g}_{i_2}^{j[\tau]}(1 - \alpha_{i_2}^{[\tau]}))\, D_{i_2}^{[\tau]} = 0 \quad \text{(A10)}$$

$$\frac{\partial L}{\partial \bar{\omega}_{i_2}^{[\tau]}} = -\frac{A_{i_2}^{[\tau]}}{(\bar{\omega}_{i_2}^{[\tau]})^2} + \frac{a_{i_2}^{[\tau]}}{\alpha_{i_2}^{[\tau]}} D_{i_2}^{[\tau]} - \sum_{j=1}^{|J|} (H^{j[\tau]} + b^{j[\tau]}) \sum_{i_2=1}^{|I_2|} (g_{i_2}^{j[\tau]}\alpha_{i_2}^{[\tau]}$$

$$+ \hat{g}_{i_2}^{j[\tau]}(1 - \alpha_{i_2}^{[\tau]}))\, D_{i_2}^{[\tau]} \frac{\partial \Delta t_i^{[\tau]}}{\partial \bar{\omega}_{i_2}^{[\tau]}}$$

$$+ \sum_{j=1}^{|J|} 0.5H^{j[\tau]} g_{i_2}^{j[\tau]}\alpha_{i_2}^{[\tau]}(1 - x'^{[\tau]}_{i_2})\, D_{i_2}^{[\tau]} + \sum_{j=1}^{|J|} 0.5H^{j[\tau]}\hat{g}_{i_2}^{j[\tau]}(1 - \alpha_{i_2}^{[\tau]})$$

$$\times (1 - x'^{[\tau]}_{i_2})\, D_{i_2}^{[\tau]} + \sum_{j=1}^{|J|} b^{j[\tau]} g_{i_2}^{j[\tau]}\alpha_{i_2}^{[\tau]}((1 - x'^{[\tau]}_{i_2})$$

$$+ (1 - \alpha_{i_2}^{[\tau]})\eta_{i_2}^{[\tau]})\, D_{i_2}^{[\tau]} + \sum_{j=1}^{|J|} b^{j[\tau]}\hat{g}_{i_2}^{j[\tau]}(1 - \alpha_{i_2}^{[\tau]})((1 - x'^{[\tau]}_{i_2})$$

$$+ \alpha_{i_2}^{[\tau]}\eta_{i_2}^{[\tau]})\, D_{i_2}^{[\tau]} - \sum_{j=1}^{|J|} 0.5H^{j[\tau]}(1 - {}'x_{i_2}^{[\tau]})\, f_{i_2}^{[\tau]} D_{i_2}^{[\tau]}$$

$$+ \sum_{j=1}^{|J|} \lambda^{j[\tau]} g_{i_2}^{j[\tau]}\alpha_{i_2}^{[\tau]}(1 - \alpha_{i_2}^{[\tau]})\eta_{i_2}^{[\tau]} D_{i_2}^{[\tau]}$$

$$+ \sum_{j=1}^{|J|} \lambda^{j[\tau]}\hat{g}_{i_2}^{j[\tau]}(1 - \alpha_{i_2}^{[\tau]})\alpha_{i_2}^{[\tau]}\eta_{i_2}^{[\tau]} D_{i_2}^{[\tau]} + \sum_{j=1}^{|J|} \lambda^{j[\tau]} f_{i_2}^{[\tau]}(1 - {}'x_{i_2}^{[\tau]})\, D_{i_2}^{[\tau]}$$

$$+ \sum_{j=1}^{|J|} \lambda^{j[\tau]}(g_{i_2}^{j[\tau]}\alpha_{i_2}^{[\tau]} + \hat{g}_{i_2}^{j[\tau]}(1 - \alpha_{i_2}^{[\tau]})) \frac{\partial \Delta t_i^{[\tau]}}{\partial \bar{\omega}_{i_2}^{[\tau]}} D_{i_2}^{[\tau]} = 0 \quad \text{(A11)}$$

$$\lambda^{j[\tau]}\left[V^{j[\tau]}(0) - \sum_{k=1}^{|K(j)|} D_k^{j[\tau]}\left[t_k^{j[\tau]} + \left(\frac{1}{\alpha_k^{j[\tau]}} - 1\right)\eta_k^{j[\tau]}\underline{\underline{\omega}}_k^{j[\tau]}\right]\right.$$

$$- \sum_{i_1=1}^{|I_1|} g_{i_1}^{j[\tau]} D_{i_1}^{[\tau]}\left[{}'t_{i_1}^{[\tau]} + \Delta t_{i_1}^{[\tau]} + \left(\frac{1}{\alpha_{i_1}^{[\tau]}} - 1\right)\eta_{i_1}^{[\tau]}\underline{\underline{\omega}}_{i_1}^{[\tau]}\right]$$

$$+ \sum_{i_1=1}^{|I_1|} f_{i_1}^{[\tau]} D_{i_1}^{[\tau]}\left[{}'t_{i_1}^{[\tau]} - (1 - {}'x_{i_1}^{[\tau]})\underline{\underline{\omega}}_{i_1}^{[\tau]} - \left(\frac{1}{\alpha_{i_1}^{[\tau]}} - 1\right)\eta_{i_1}^{[\tau]}\underline{\underline{\omega}}_{i_1}^{[\tau]}\right]$$

$$- \sum_{i_2=1}^{|I_2|} g_{i_2}^{j[\tau]}\alpha_{i_2}^{[\tau]} D_{i_2}^{[\tau]}[{}'t_{i_2}^{[\tau]} + \Delta t_{i_2}^{[\tau]} + (1 - \alpha_{i_2}^{[\tau]})\eta_{i_2}^{[\tau]}\bar{\omega}_{i_2}^{[\tau]}] - \sum_{i_2=1}^{|I_2|} \hat{g}_{i_2}^{j[\tau]}$$

$$\times (1 - \alpha_{i_2}^{[\tau]})\, D_{i_2}^{[\tau]}[{}'t_{i_2}^{[\tau]} + \Delta t_{i_2}^{[\tau]} + \alpha_{i_2}^{[\tau]}\eta_{i_2}^{[\tau]}\bar{\omega}_{i_2}^{[\tau]}] + \sum_{i_2=1}^{|I_2|} f_{i_2}^{[\tau]} D_{i_2}^{[\tau]}[{}'t_{i_2}^{[\tau]}$$

$$- (1 - {}'x_{i_2}^{[\tau]})\bar{\omega}_{i_2}^{[\tau]}] + \sum_{m=1}^{|M(j)|} D_m^{j[\tau]}[t_m^{j[\tau]} - (1 - x_m^{j[\tau]})\underline{\underline{\omega}}_m^{j[\tau]} - d_m^{j[\tau]}]$$

$$\left. + \sum_{n=1}^{|N(j)|} D_n^{j[\tau]}\left[t_n^{j[\tau]} - (1 - x_n^{j[\tau]})\underline{\underline{\omega}}_n^{j[\tau]} - \left(\frac{1}{\alpha_n^{j[\tau]}} - 1\right)\eta_n^{j[\tau]}\underline{\underline{\omega}}_n^{j[\tau]}\right]\right] = 0 \quad \text{(A12)}$$

$$\lambda^{j[\tau]} = H^{j[\tau]} + b^{j[\tau]} \quad \text{(A13)}$$

Solving Eqs. A5, A7, A9, and A11 with Eq. A13 gives Eqs. 25, 27, 29, and 31 in the main text. The solution of Eq. A12 gives Eq. 33 in the main text.

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